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PURE MATHEMATICS FOR PRE-BEGINNERS

Logic
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By Dr. Steve Warner

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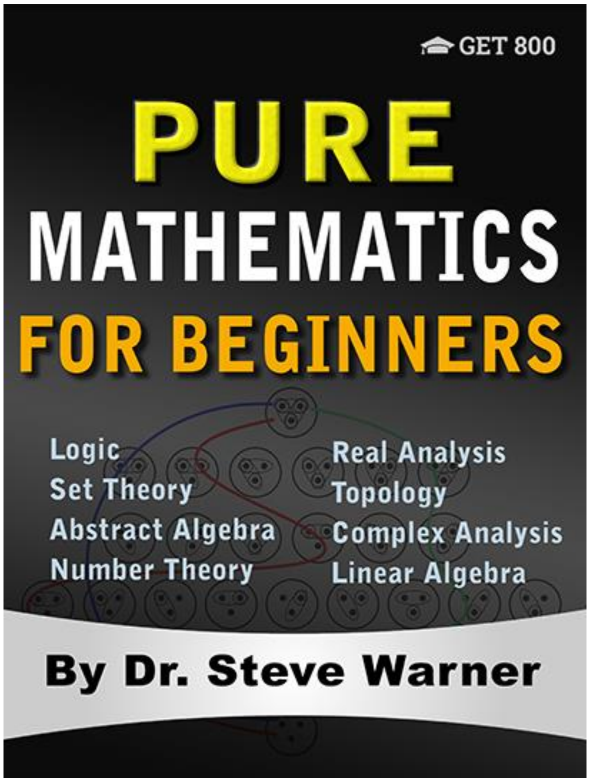

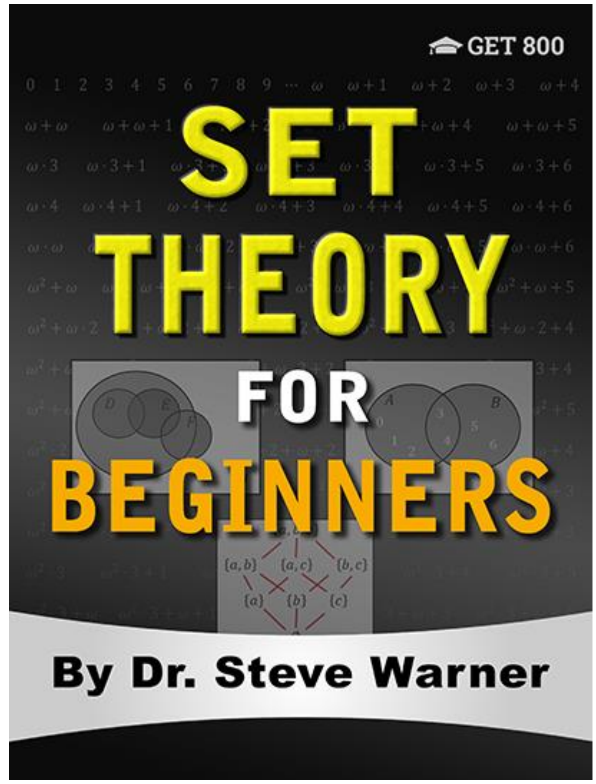

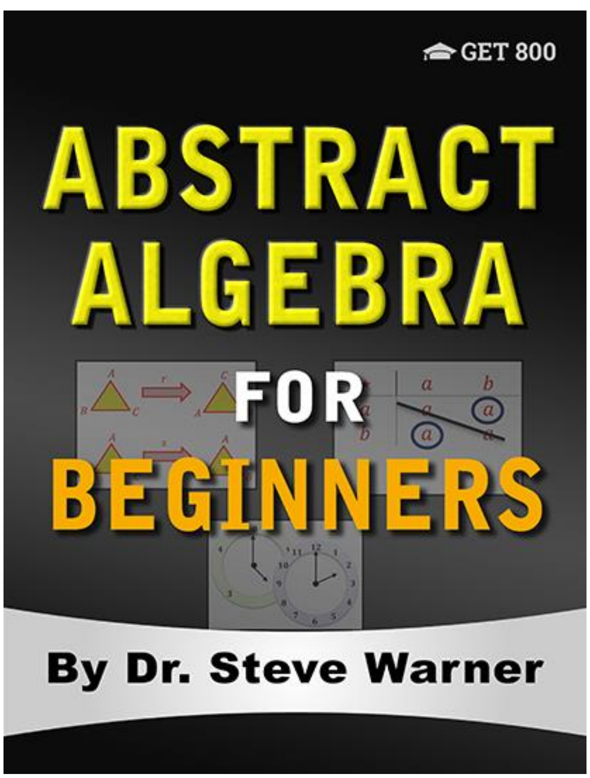

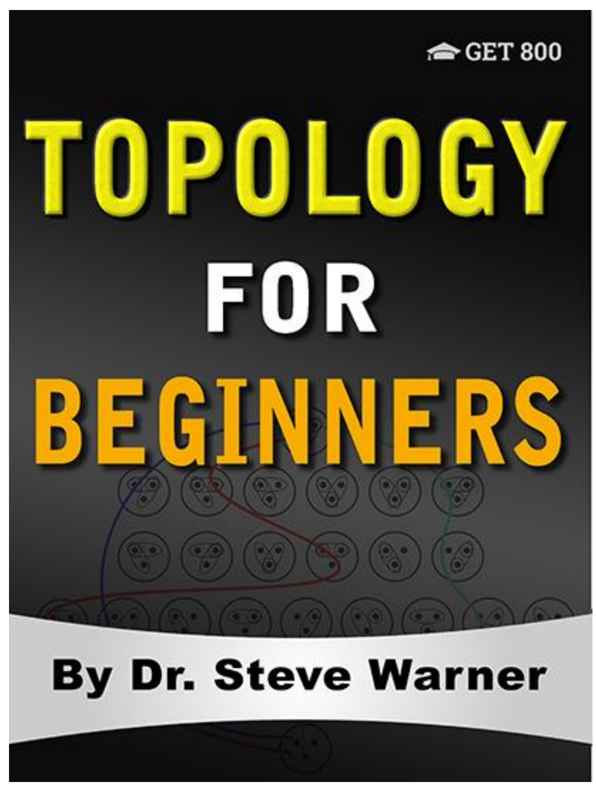

Pure Mathematics for Pre-Beginners

An Elementary Introduction to Logic, Set Theory, Abstract Algebra, Number Theory, Real Analysis, Topology, Complex Analysis, and Linear Algebra

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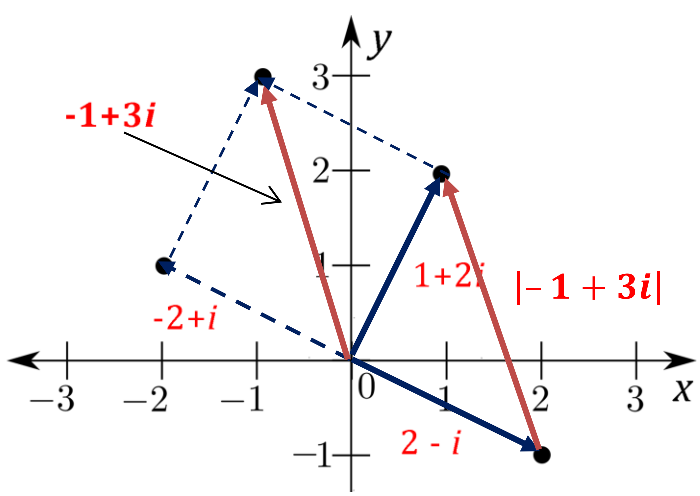
Dr. Steve Warner



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INTRODUCTION

PURE MATHEMATICS FOR PRE-BEGINNERS

Shortly after the release of *Pure Mathematics for Beginners,* I began receiving messages from a wide range of readers that were enjoying the book. These readers include:

- students struggling in their advanced math and computer science classes,
- physicists that wanted to learn some theoretical mathematics,
- high school math teachers that wanted to introduce their more advanced math students to mathematical theory,
- and many others.

However, I quickly learned that the book was too difficult for some readers. I refer to these readers as “pre-beginners.” By a “pre-beginner,” I mean a student that is ready to start learning some more advanced mathematics, but is not quite ready to dive into proofwriting.

The book you are currently reading was written with these “pre-beginners” in mind. It provides a more basic ***nonrigorous*** introduction to pure mathematics, while exposing readers to a wide range of mathematical topics in logic, set theory, abstract algebra, number theory, real analysis, topology, complex analysis, and linear algebra.

There are no prerequisites for this book. The content is completely self-contained. Furthermore, reading this book will naturally increase a student’s level of “mathematical maturity.” Although there is no single agreed upon definition of mathematical maturity, one reasonable way to define it is as “one’s ability to analyze, understand, and communicate mathematics.” A student with a very high level of mathematical maturity may find this book very easy—this student may want to go through the book quickly and then move on to *Pure Mathematics for Beginners*. A student with a lower level of mathematical maturity will probably find this book more challenging. However, the reward will certainly be more than worth the effort.

If you read this book and complete the exercises along the way, then your level of mathematical maturity will continually be increasing. This increased level of mathematical maturity will not only help you to succeed in advanced math courses, but it will improve your general problem solving and reasoning skills. This will make it easier to improve your performance in college, in your professional life, and on standardized tests such as the SAT, ACT, GRE, and GMAT.

At the end of each lesson there is a Problem Set. The problems in each of these Problem Sets have been organized into five levels of difficulty, followed by several Challenge Problems. Level 1 problems are the easiest and Level 5 problems are the most difficult, except for the Challenge Problems. If you want to get just a small taste of pure mathematics, then you can work on the easier problems. If you want to achieve a deeper understanding of the material, take some time to struggle with the harder problems.

The author welcomes all feedback. Feel free to email Dr. Steve Warner with any questions and comments at steve@SATPrepGet800.com.

LESSON 1
LOGIC

Statements

In mathematics, a **statement** (or **proposition**) is a sentence that can be true or false, but not both simultaneously.

Example 1.1: "Jennifer is working" is a statement because at any given time either Jennifer is working or Jennifer is not working.

Example 1.2: The sentence "Go away!" is ***not*** a statement because it cannot be true or false. This sentence is a **command**.

Exercise 1.3: Determine if each of the following sentences are statements:

1. The check is in the mail. ______________________________
2. Are you feeling okay? ______________________________
3. Unicorns are real. ______________________________
4. Don't count your chickens before they hatch. ______________________________
5. Odin is chasing a mouse. ______________________________

An **atomic statement** expresses a single idea. The statement "Jennifer is working" that we discussed above is an example of an atomic statement. Let's look at a few more examples.

Example 1.4: The following sentences are atomic statements:

1. 26 is an even number.
2. Mary Tudor was the first queen of England.
3. $8 < 2$.
4. Gregory is 1.8 meters tall.
5. Life exists on planets other than the Earth.

Notes: Sentences 1 and 2 above are true atomic statements and sentence 3 is a false atomic statement.

We can't say for certain whether sentence 4 is true or false without knowing who Gregory is. However, it is either true or false. Therefore, it is a statement. Since it expresses a single idea, it is an atomic statement.

It is also unknown whether sentence 5 is true or false, but this does not change the fact that it must be either true or false. Furthermore, it expresses a single idea. Therefore, it is an atomic statement.

We use **logical connectives** to form **compound statements**. The most commonly used logical connectives are "and," "or," "if…then," "if and only if," and "not."

Example 1.5: The following sentences are compound statements:

1. 26 is an even number and $0 = 1$.
2. Ahmed is holding a piece of chalk or soda is a beverage.
3. If Odin is a cat, then elephants can breathe underwater.
4. Abraham Lincoln is alive today if and only if $5 - 3 = 2$.
5. 11 is not a prime number.

Sentence 1 above uses the logical connective "and." Since the statement "$0 = 1$" is false, it follows that sentence 1 is false. It does not matter that the statement "26 is an even number" is true. In fact, "T and F" is always F.

Sentence 2 uses the logical connective "or." Since the statement "soda is a beverage" is true, it follows that sentence 2 is true. It does not even matter whether Ahmed is holding a piece of chalk. In fact, "T or T" is always true and "F or T" is always T.

It's worth pausing for a moment to note that in the English language the word "or" has two possible meanings. There is an "inclusive or" and an "exclusive or." The "inclusive or" is true when both statements are true, whereas the "exclusive or" is false when both statements are true. In mathematics, by default, we always use the "inclusive or" unless we are told to do otherwise. To some extent, this is an arbitrary choice that mathematicians have agreed upon. However, it can be argued that it is the better choice since it is used more often and it is easier to work with. Note that we were assuming use of the "inclusive or" in the last paragraph when we said, "In fact, "T or T" is always true." See Problems 25 and 26 below for more on the "exclusive or."

Sentence 3 uses the logical connective "if…then." The statement "elephants can breathe underwater" is false. We need to know whether Odin is a cat in order to figure out the truth value of sentence 3. If Odin is a cat, then sentence 3 is false ("if T, then F" is always F). If Odin is not a cat, then sentence 3 is true ("if F, then F" is always T). Do not worry if you are confused about where the truth values just mentioned come from. We will discuss the logical connective "if…then" (as well as all the other connectives) in much more detail in the section on logical connectives below.

Sentence 4 uses the logical connective "if and only if." Since the two atomic statements have different truth values, it follows that sentence 4 is false. In fact, "F if and only if T" is always F.

Sentence 5 uses the logical connective "not." Since the statement "11 is a prime number" is true, it follows that sentence 5 is false. In fact, "not T" is always F.

Notes: (1) The logical connectives "and," "or," "if…then," and "if and only if," are called **binary connectives** because they join two statements (the prefix "bi" means "two").

(2) The logical connective "not" is called a **unary connective** because it is applied to just a single statement ("unary" means "acting on a single element").

(3) Don't worry if the meaning of any of these logical connectives confuses you. We will learn more about them in the section on logical connectives below.

Exercise 1.6: Determine if each of the following statements is an atomic statement or a compound statement.

1. Silence makes me angry. ____________________
2. We believe in this or we believe in that. ____________________
3. Charles Darwin did not believe in evolution. ____________________
4. My girlfriend watches the television show Rick and Morty. ____________________
5. If dragons are real, then I am a dragon. ____________________
6. Sentences that begin with the word "why" are questions. ____________________
7. You must find your keys or you will not be on time. ____________________
8. The universe will eventually collapse if and only if it is currently expanding. ____________________
9. The word "and" has the same meaning as the word "or." ____________________
10. John likes to walk, but he doesn't like to run. ____________________

Example 1.7: The following sentences are ***not*** statements:

1. When will you be back?
2. Leave me alone!
3. $x + 1 = 3$
4. This sentence is false.
5. This sentence is true.

Sentence 1 above is a question and sentence 2 is a command.

Sentence 3 has an unknown variable – it can be turned into a statement by assigning a value to the variable.

Sentences 4 and 5 are self-referential (they refer to themselves). They can be neither true nor false. Sentence 4 is called the Liar's paradox and sentence 5 is called a vacuous affirmation.

Truth Assignments

We will use letters such as p, q, r, and s to denote atomic statements. We will refer to these letters as **propositional variables**, and we will generally assign a truth value of T (for true) or F (for false) to each propositional variable. Formally, we define a **truth assignment** of a list of propositional variables to be a choice of T or F for each propositional variable in the list.

Example 1.8: Consider the propositional variable p. There are **two** possible truth assignments for this propositional variable as follows:

1. We can assign p to be true.
2. We can assign p to be false.

We can visualize this list of truth assignments with the following table:

p
T
F

Observe how the table has just one column because there is only one propositional variable. We label the column with the propositional variable p. Underneath the propositional variable, we have two rows—one for each of the two possible truth assignments.

Example 1.9: Consider the propositional variables p and q (where p and q are **distinct**). There are **four** possible truth assignments for this list of propositional variables as follows:

1. We can assign both p and q to be true.
2. We can assign p to be true and q to be false.
3. We can assign p to be false and q to be true.
4. We can assign both p and q to be false.

Note: When we say that p and q are **distinct**, we mean that $p \neq q$. In other words, we use the word distinct when we want to make sure that it is understood that the objects under consideration are different from each other.

We can visualize this list of truth assignments with the following table:

p	q
T	T
T	F
F	T
F	F

Observe how the table has two columns because there are two propositional variables. We label the columns with the propositional variables p and q. Underneath the propositional variables, we have four rows—one for each of the four possible truth assignments.

Exercise 1.10: Consider the three distinct propositional variables p, q, and r. How many different truth assignments are there for this list of propositional variables? ______________________________

Draw a table that will allow us to visualize this list of truth assignments.

Logical Connectives

We use the symbols $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and $\neg$ for the most common logical connectives. The truth value of a compound statement is determined by the truth values of its atomic parts together with applying various rules for the connectives. Let's look at each connective in detail.

We will use the "wedge" symbol $\wedge$ to represent the logical connective "and." The compound statement $p \wedge q$ is called the **conjunction** of p and q. It is pronounced "$\boldsymbol{p}$ **and** $\boldsymbol{q}$." $p \wedge q$ is true when both p and q are true, and it is false otherwise.

The following table summarizes the truth values of $p \wedge q$ for each possible truth assignment of the propositional variables p and q.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Notes: (1) The table displayed above is called a **truth table**. This type of table is used to display the possible truth values of a compound statement. We start by labelling the columns of the table with the propositional variables that appear in the statement, followed by the statement itself. We then use the rows to run through every possible combination of truth values for the propositional variables (all possible truth assignments) followed by the resulting truth values for the compound statement.

(2) The first two columns of the truth table above (labeled p and q) give the four possible truth assignments for the propositional variables p and q (see Example 1.9).

(3) The truth table for the conjunction is based upon the way we use the word "and" in everyday English. For example, suppose that Jamie is a girl with black hair. Then the statement "Jamie is a girl and Jamie has black hair" is true because each of the statements "Jamie is a girl" and "Jamie has black hair" are true. Similarly, the statement "Jamie is a girl and Jamie has red hair" is false because the statement "Jamie has red hair" is false. Based upon your own experience of the English language, you may be able to compute the truth value of a conjunction of two statements without needing to look back at the truth table.

Example 1.11: If p is true and q is false, then we can compute the truth value of $p \wedge q$ by looking at the second row of the truth table for the conjunction.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \wedge q}$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

We see from the highlighted row in the truth table above that $p \wedge q \equiv \mathrm{T} \wedge \mathrm{F} \equiv \mathbf{F}$.

Note: Here the symbol $\equiv$ can be read "is logically equivalent to." So, we see that if p is true and q is false, then $p \wedge q$ is logically equivalent to F, or more simply, $p \wedge q$ is false.

Exercise 1.12: Determine the truth value of $p \wedge q$ given that

1. p and q are both true. ___
2. p and q are both false. ___
3. p is false and q is true. ___

We will use the "vee" symbol $\vee$ to represent the logical connective "or." The compound statement $p \vee q$ is called the **disjunction** of p and q. It is pronounced "$\boldsymbol{p}$ **or** $\boldsymbol{q}$." $p \vee q$ is true when p or q (or both) are true, and it is false when p and q are both false.

The following table summarizes the truth values of $p \vee q$ for each possible truth assignment of the propositional variables p and q.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \vee q}$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Notes: (1) The truth table for the disjunction is based upon the way we use the "inclusive or" in everyday English. For example, suppose that in order to be able to watch television, Kelly's parents tell her that she must first either do the dishes or clean her room. In this context, the statement "Kelly does the dishes or Kelly cleans her room" is understood to be true if Kelly does the dishes or Kelly cleans her room **or both**. Certainly if she completes both of these tasks, it would be unnatural to penalize her.

(2) In mathematics, when we use the word "or" we always assume that we mean the "inclusive or" unless we are told otherwise. In English when we use the word "or," we are more likely to be using the "exclusive or." For example, if a waiter says "you can have fries or a salad with your order," it is unlikely that he means you can have both. Indeed, this is an example of the "exclusive or." See Problems 25 and 26 below for more on the exclusive or.

Exercise 1.13: Determine the truth value of $p \vee q$ given that

1. p and q are both true. ___
2. p and q are both false. ___
3. p is true and q is false. ___
4. p is false and q is true. ___

We will use the "rightarrow" symbol $\rightarrow$ to represent the logical connective "if...then." The compound statement $p \rightarrow q$ is called a **conditional** or **implication**. It is pronounced "**if** $\boldsymbol{p}$**, then** $\boldsymbol{q}$" or $\boldsymbol{p}$ **implies** $\boldsymbol{q}$. $p \rightarrow q$ is true when p is false or q is true (or both), and it is false when p is true and q is false.

The following table summarizes the truth values of $p \to q$ for each possible truth assignment of the propositional variables p and q.

p	q	$p \to q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Notes: (1) In the conditional $p \to q$, p is called the **hypothesis** (or **assumption** or **premise**) and q is called the **conclusion**.

(2) The truth table for the conditional is loosely based upon the English meaning of "if…then" or "implies." However, trying to understand this truth table by analyzing English sentences often leads to confusion. So, in this case, it may be more instructive to understand what we wish to accomplish with this connective mathematically. When should $p \to q$ be true. Well, we would like $p \to q$ to be true if the assumption that p is true forces q to be true as well (equivalently, if the hypothesis is true, then the conclusion must be true). For example, let's take the statement "If Odin is a cat, then Odin can bark." Now, if Odin happens to be a cat, then the statement just given in quotes is false. Do you see why? The hypothesis "Odin is a cat" is true. If the conditional were true, then the conclusion "Odin can bark" would be forced to be true. However, cats can't bark. So, the conditional is false. This situation corresponds to the second row in the truth table for the conditional above.

On the other hand, the statement "If Odin is a cat, then Odin can meow" is most likely true. This time the hypothesis and conclusion are both true. This situation corresponds to the first row in the truth table for the conditional above.

(3) What about the situation in which the hypothesis is false. In this case, we don't really care what the conclusion is. The conditional is true either way. When the hypothesis is false, we will say that the conditional statement is **vacuously true**. The word "vacuous" means "empty." The idea is that something that is vacuously true is true for a very silly reason. For example, suppose that we are looking at an empty room and someone says, "If there is a pig in that room, then it can fly." This is equivalent to saying, "Every pig in that room can fly." This statement is true, but for a very dumb reason. Yes, every pig in that room can fly simply because there are no pigs in the room. If there were even a single pig in the room, then the statement would be false. In order for someone to dispute our claim that every pig in the room can fly, they would need to show us a pig in the room that cannot fly. Of course, they cannot do this. After all, there are no pigs in the room. This notion of "vacuous truth" corresponds to the third and fourth rows in the truth table for the conditional above.

Exercise 1.14: Determine the truth value of $p \to q$ given that

1. p and q are both true. ___
2. p and q are both false. ___
3. p is true and q is false. ___
4. p is false and q is true. ___

We will use the "doublearrow" symbol $\leftrightarrow$ to represent the logical connective "if and only if." The compound statement $p \leftrightarrow q$ is called a **biconditional**. It is pronounced "$\boldsymbol{p}$ **if and only if** $\boldsymbol{q}$." $p \leftrightarrow q$ is true when p and q have the same truth value (both true or both false), and it is false when p and q have opposite truth values (one true and the other false).

The following table summarizes the truth values of $p \leftrightarrow q$ for each possible truth assignment of the propositional variables p and q.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p \leftrightarrow q}$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

Exercise 1.15: Determine the truth value of $p \leftrightarrow q$ given that

1. p and q are both true. ___
2. p and q are both false. ___
3. p is true and q is false. ___
4. p is false and q is true. ___

We will use the "taildash" symbol $\neg$ to represent the logical connective "not." The compound statement $\neg p$ is called the **negation** of p. It is pronounced "**not** $\boldsymbol{p}$." $\neg p$ is true when p is false, and it is false when p is true (p and $\neg p$ have opposite truth values.)

The following table summarizes the truth values of $\neg p$ for each possible truth assignment of the propositional variable p.

$\boldsymbol{p}$	$\boldsymbol{\neg p}$
T	**F**
F	**T**

Notes: (1) Since negation requires only a single propositional variable, there are just two possible truth assignments to worry about. The first column of the truth table above (labeled p) gives the two possible truth assignments (T and F).

(2) The truth table for the negation is based upon the way we use the word "not" in everyday English. For example, since the statement "Fish swim" is true, it follows that the statement "Fish do not swim" is false. Similarly, since the statement "Elephants fly" is false, it follows that the statement "Elephants do not fly" is true.

Example 1.16: If p is true, then we can compute the truth value of $\neg p$ by looking at the first row of the truth table for the negation.

p	$\neg p$
T	**F**
F	**T**

We see from the highlighted row in the truth table above that $\neg p \equiv \neg \mathrm{T} \equiv \mathbf{F}$.

Exercise 1.17: Determine the truth value of $\neg p$ given that p is false. ___

Example 1.18: Let p represent the statement "Ducks quack" and let q represent the statement "$0 = 1$." Note that p is true and q is false.

1. $p \wedge q$ represents "Ducks quack and $0 = 1$." Since q is false, it follows that $p \wedge q$ is false.
2. $p \vee q$ represents "Ducks quack or $0 = 1$." Since p is true, it follows that $p \vee q$ is true.
3. $p \to q$ represents "If ducks quack, then $0 = 1$." Since p is true and q is false, $p \to q$ is false.
4. $p \leftrightarrow q$ represents "Ducks quack if and only if $0 = 1$." Since p is true and q is false, $p \leftrightarrow q$ is false.
5. $\neg q$ represents the statement "$0 \neq 1$." Since q is false, $\neg q$ is true.
6. $\neg p \vee q$ represents the statement "Ducks don't quack or $0 = 1$." Since $\neg p$ and q are both false, $\neg p \vee q$ is false. Note that $\neg p \vee q$ always means $(\neg p) \vee q$. In general, without parentheses present, we always apply negation before any of the other connectives.
7. $\neg(p \vee q)$ represents the statement "It is not the case that either ducks quack or $0 = 1$." This can also be stated as "Neither do ducks quack nor is 0 equal to 1." Since $p \vee q$ is true (see 2 above), $\neg(p \vee q)$ is false.
8. $\neg p \wedge \neg q$ represents the statement "Ducks don't quack and $0 \neq 1$." This statement can also be stated as "Neither do ducks quack nor is 0 equal to 1." Since this is the same statement as in 7 above, it should follow that $\neg p \wedge \neg q$ is equivalent to $\neg(p \vee q)$. You will be asked to verify this later (see Exercise 1.27 below). For now, let's observe that since $\neg p$ is false, it follows that $\neg p \wedge \neg q$ is false. This agrees with the truth value we got in 7.

Note: The equivalence of $\neg(p \vee q)$ with $\neg p \wedge \neg q$ (see parts 7 and 8 of Example 1.18 above) is one of **De Morgan's laws**. These laws will be explored further below (see Example 1.26 and Exercise 1.27).

Exercise 1.19: Let p represent the statement "Frogs are birds," and let q represent the statement "$2 < 1$." Translate each of the following compound statements into English and compute the truth value of each statement.

1. $p \to q$ ______________________________
2. $\neg p \vee q$ ______________________________
3. $p \leftrightarrow q$ ______________________________
4. $(p \to q) \wedge (q \to p)$ ______________________________
5. $\neg(p \wedge q)$ ______________________________
6. $\neg p \vee \neg q$ ______________________________

Evaluating Truth

Example 1.20: Let p, q, and r be propositional variables with p and q true, and r false. Let's compute the truth value of $\neg p \vee (\neg q \to r)$.

Truth table solution: One foolproof way to compute the desired truth value is to build the whole truth table of $\neg p \vee (\neg q \to r)$ one column at a time. Since there are 3 propositional variables (p, q, and r), we will need 8 rows to get all the possible truth values (see Exercise 1.10 and its solution). We then create a column for each compound statement that appears within the given statement starting with the statements of smallest length and working our way up to the given statement. We will need columns for p, q, r (the atomic statements), $\neg p$, $\neg q$, $\neg q \to r$, and finally, the statement itself, $\neg p \vee (\neg q \to r)$. Below is the final truth table with the relevant row highlighted and the final answer circled.

p	q	r	$\neg p$	$\neg q$	$\neg q \to r$	$\neg p \vee (\neg q \to r)$
T	T	T	F	F	T	T
T	T	F	F	F	T	T
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Notes: (1) We fill out the first three columns of the truth table by listing all possible combinations of truth assignments for the propositional variables p, q, and r. Notice how down the first column we have 4 T's followed by 4 F's, down the second column we alternate sequences of 2 T's with 2 F's, and down the third column we alternate T's with F's one at a time. This is a nice systematic way to make sure we get all possible combinations of truth assignments.

If you're having trouble seeing the pattern of T's and F's, here is another way to think about it: In the first column, the first half of the rows have a T and the remainder have an F. This gives 4 T's followed by 4 F's.

For the second column, we take half the number of consecutive T's in the first column (half of 4 is 2) and then we alternate between 2 T's and 2 F's until we fill out the column.

For the third column, we take half the number of consecutive T's in the second column (half of 2 is 1) and then we alternate between 1 T and 1 F until we fill out the column.

(2) Since the connective $\neg$ has the effect of taking the opposite truth value, we generate the entries in the fourth column by taking the opposite of each truth value in the first column. Similarly, we generate the entries in the fifth column by taking the opposite of each truth value in the second column.

(3) For the sixth column, we apply the connective $\to$ to the fifth and third columns, respectively, and finally, for the last column, we apply the connective $\vee$ to the fourth and sixth columns, respectively.

(4) The original question is asking us to compute the truth value of $\neg p \vee (\neg q \to r)$ when p and q are true, and r is false. In terms of the truth table, we are being asked for the entry in the second row and last (seventh) column. Therefore, the answer is $\mathbf{T}$.

(5) This is certainly not the most efficient way to answer the given question. However, building truth tables is not too difficult, and it's a foolproof way to determine truth values of compound statements.

Alternate Solution: We have $\neg p \vee (\neg q \to r) \equiv \neg \mathrm{T} \vee (\neg \mathrm{T} \to \mathrm{F}) \equiv \mathrm{F} \vee (\mathrm{F} \to \mathrm{F}) \equiv \mathrm{F} \vee \mathrm{T} \equiv \mathbf{T}$.

Notes: (1) For the first equivalence, we simply replaced the propositional variables by their given truth values. We replaced p and q by T, and we replaced r by F.

(2) For the second equivalence, we used the first row of the truth table for the negation (drawn to the right for your convenience).

p	$\neg p$
T	**F**
F	**T**

We see from the highlighted row that $\neg \mathrm{T} \equiv \mathrm{F}$. We applied this result twice.

(3) For the third equivalence, we used the fourth row of the truth table for the conditional.

p	q	$p \to q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

We see from the highlighted row that $\mathrm{F} \to \mathrm{F} \equiv \mathrm{T}$.

(4) For the last equivalence, we used the third row of the truth table for the disjunction.

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

We see from the highlighted row that $\mathrm{F} \vee \mathrm{T} \equiv \mathrm{T}$.

(5) We can save a little time by immediately replacing the negation of a propositional variable by its truth value (which will be the opposite truth value of the propositional variable). For example, since p has truth value T, we can replace $\neg p$ by F. Similarly, since q has truth value T, we can replace $\neg q$ by F. The faster solution would look like this:

$$\neg p \vee (\neg q \to r) \equiv \mathrm{F} \vee (\mathrm{F} \to \mathrm{F}) \equiv \mathrm{F} \vee \mathrm{T} \equiv \mathbf{T}.$$

Quicker solution: Since q has truth value T, it follows that $\neg q$ has truth value F. So, $\neg q \to r$ has truth value T. Finally, $\neg p \vee (\neg q \to r)$ must then have truth value T.

Notes: (1) Symbolically, we can write the following:

$$\neg p \vee (\neg q \to r) \equiv \neg p \vee (\neg \mathrm{T} \to r) \equiv \neg p \vee (\mathrm{F} \to r) \equiv \neg p \vee \mathrm{T} \equiv \mathbf{T}$$

(2) We can display this reasoning visually as follows:

$$\begin{array}{ccccccc}
\neg p & \vee & (& \neg & q & \to & r) \\
 & & & & \mathrm{T} & & \\
 & & & \mathrm{F} & & & \\
 & & & & & \mathrm{T} & \\
 & \mathbf{T} & & & & &
\end{array}$$

The vertical lines have just been included to make sure you see which connective each truth value is written below.

We began by placing a T under the propositional variable q to indicate that q is true. Since $\neg \mathrm{T} \equiv \mathrm{F}$, we then place an F under the negation symbol. Next, since $\mathrm{F} \to r \equiv \mathrm{T}$ regardless of the truth value of r, we place a T under the conditional symbol. Finally, since $\neg p \vee \mathrm{T} \equiv \mathrm{T}$ regardless of the truth value of p, we place a T under the disjunction symbol. We made this last T bold to indicate that we are finished.

(3) Knowing that q has truth value T is enough to determine the truth value of $\neg p \vee (\neg q \to r)$, as we saw in Note 1 above. It's okay if you didn't notice that right away. This kind of reasoning takes a bit of practice and experience.

Exercise 1.21: Let p, q, and r be propositional variables.

1. Draw the truth table for $p \leftrightarrow (q \wedge \neg r)$.

2. Use the truth table from part 1 to compute the truth value of $p \leftrightarrow (q \wedge \neg r)$ when p is true, and q and r are false. ____

3. Suppose that p and r are both true. Is this enough information to compute the truth value of $p \leftrightarrow (q \wedge \neg r)$? _____ If so, what is that truth value? ______________________________

Logical Equivalence

We say that two statements are **logically equivalent** if they have the same truth table. We use the symbol "≡" to indicate logical equivalence.

Example 1.22: Let p be a propositional variable. Let's show that p and $\neg(\neg p)$ are logically equivalent (symbolically, we write $p \equiv \neg(\neg p)$). We will show that p and $\neg(\neg p)$ have the same truth table. We can put all the information into a single table.

$\boldsymbol{p}$	$\boldsymbol{\neg p}$	$\boldsymbol{\neg(\neg p)}$
T	**F**	**T**
F	**T**	**F**

Observe that the first column gives the truth values for p, the third column gives the truth values for $\neg(\neg p)$, and both these columns are identical. It follows that $p \equiv \neg(\neg p)$.

Notes: (1) The logical equivalence $p \equiv \neg(\neg p)$ is called the **law of double negation**. In words, this law says that if you negate a propositional variable twice, the resulting statement is logically equivalent to the original propositional variable.

(2) As an example in English, let p be the statement "John is hungry." Then the statement $\neg(\neg p)$ can be expressed in English as "It is not the case that John is not hungry." By the law of double negation, these two statements are logically equivalent.

Exercise 1.23: The **law of the conditional** is the logical equivalence $p \to q \equiv \neg p \vee q$. Use a truth table to verify this logical equivalence.

Note: The law of the conditional allows us to replace the conditional statement $p \to q$ by the more intuitive statement $\neg p \vee q$. Remember that we can think of the conditional statement $p \to q$ as having the hypothesis p and the conclusion q. The disjunctive form $\neg p \vee q$ tells us quite explicitly that a conditional statement is true precisely if the hypothesis p is false or the conclusion q is true (or both).

Consider the conditional statement $p \to q$. There are three other statements associated with this statement.

1. The **converse** is the statement $q \to p$.
2. The **inverse** is the statement $\neg p \to \neg q$.
3. The **contrapositive** is the statement $\neg q \to \neg p$.

Example 1.24: Consider the conditional statement "If you are a cat, then you are a mammal."

1. The converse is the statement "If you are a mammal, then you are a cat."
2. The inverse is the statement "If you are not a cat, then you are not a mammal."
3. The contrapositive is the statement "If you are not a mammal, then you are not a cat."

Notes: (1) If we let p be the statement "You are a cat" and we let q be the statement "You are a mammal," then the given conditional statement can be represented by $p \to q$. Similarly, the converse can be represented by $q \to p$, the inverse can be represented by $\neg p \to \neg q$, and the contrapositive can be represented by $\neg q \to \neg p$.

(2) Notice that in this example, the given conditional statement is true. Indeed, every cat is a mammal. On the other hand, the converse is false. After all, there are certainly mammals that are not cats. For example, a dog is a mammal that is not a cat. This example shows that a conditional statement is **not** logically equivalent to its converse.

(3) In this example, the inverse is also false. For example, a dog is not a cat, and yet, a dog is a mammal. This example shows that a conditional statement is **not** logically equivalent to its inverse.

(4) In this example, the contrapositive is true. Anything that is not a mammal cannot possibly be a cat. In fact, it turns out that a conditional statement is **always** logically equivalent to its contrapositive. We will see this in Example 1.25 below. A word of caution is in order here. The one example just given does **not** prove that the given conditional statement is logically equivalent to its contrapositive. To verify logical equivalence, we need to check the whole truth table.

Exercise 1.25: The **law of the contrapositive** is the logical equivalence $p \to q \equiv \neg q \to \neg p$. Use a truth table to verify this logical equivalence.

The **De Morgan's laws** provide formulas for negating a conjunction and for negating a disjunction.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad\qquad \neg(p \vee q) \equiv \neg p \wedge \neg q$$

Example 1.26: Let's verify the first De Morgan's law. In other words, we will show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent. We will provide two different methods.

Direct method: If $p \equiv \mathrm{F}$ or $q \equiv \mathrm{F}$, then $\neg(p \wedge q) \equiv \neg\mathrm{F} \equiv \mathrm{T}$ and $\neg p \vee \neg q \equiv \mathrm{T}$ (because $\neg p \equiv \mathrm{T}$ or $\neg q \equiv \mathrm{T}$). If $p \equiv \mathrm{T}$ and $q \equiv \mathrm{T}$, then $\neg(p \wedge q) \equiv \neg\mathrm{T} \equiv \mathrm{F}$ and $\neg p \vee \neg q \equiv \mathrm{F} \vee \mathrm{F} \equiv \mathrm{F}$. So, all four possible truth assignments of p and q lead to the same truth value for $\neg(p \wedge q)$ and $\neg p \vee \neg q$. It follows that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Truth table method:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Observe that the sixth column gives the truth values for $\neg(p \wedge q)$, the seventh column gives the truth values for $\neg p \vee \neg q$, and both these columns are identical. It follows that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Exercise 1.27: Use a truth table to verify the second De Morgan's law $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

List 1.28: Here is a list of some useful logical equivalences. The dedicated reader may want to verify each of these by drawing a truth table or by using direct arguments similar to that used in Example 1.26 (some of these are asked for in Problems 57 through 62 below).

1. **Law of double negation:** $p \equiv \neg(\neg p)$
2. **De Morgan's laws:** $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\quad \neg(p \vee q) \equiv \neg p \wedge \neg q$
3. **Commutative laws:** $p \wedge q \equiv q \wedge p$ $\quad p \vee q \equiv q \vee p$
4. **Associative laws:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $\quad (p \vee q) \vee r \equiv p \vee (q \vee r)$
5. **Distributive laws:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $\quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6. **Identity laws:** $p \wedge \mathrm{T} \equiv p$ $\quad p \wedge \mathrm{F} \equiv \mathrm{F}$ $\quad p \vee \mathrm{T} \equiv \mathrm{T}$ $\quad p \vee \mathrm{F} \equiv p$
7. **Negation laws:** $p \wedge \neg p \equiv \mathrm{F}$ $\quad p \vee \neg p \equiv \mathrm{T}$
8. **Redundancy laws:** $p \wedge p \equiv p$ $\quad p \vee p \equiv p$
9. **Absorption laws:** $(p \vee q) \wedge p \equiv p$ $\quad (p \wedge q) \vee p \equiv p$
10. **Law of the conditional:** $p \to q \equiv \neg p \vee q$
11. **Law of the contrapositive:** $p \to q \equiv \neg q \to \neg p$
12. **Law of the biconditional:** $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Note: Although this is a fairly long list of laws, a lot of it is quite intuitive. For example, in English the word "and" is commutative. The statements "I have a cat and I have a dog" and "I have a dog and I have a cat" have the same meaning. So, it's easy to see that $p \wedge q \equiv q \wedge p$ (the first law in 3 above). As another example, the statement "I have a cat and I do not have a cat" could never be true. So, it's easy to see that $p \wedge \neg p \equiv \mathrm{F}$ (the first law in 7 above).

Example 1.29: Let's show that the statement $p \wedge [(p \wedge \neg q) \vee q]$ is logically equivalent to the atomic statement p.

Solution:

$$p \wedge [(p \wedge \neg q) \vee q] \equiv p \wedge [q \vee (p \wedge \neg q)] \equiv p \wedge [(q \vee p) \wedge (q \vee \neg q)] \equiv p \wedge [(q \vee p) \wedge \mathrm{T}]$$
$$\equiv p \wedge (q \vee p) \equiv (q \vee p) \wedge p \equiv (p \vee q) \wedge p \equiv p$$

So, we see that $p \wedge [(p \wedge \neg q) \vee q]$ is logically equivalent to the atomic statement p.

Notes: (1) For the first equivalence, we used the second commutative law.

(2) For the second equivalence, we used the second distributive law.

(3) For the third equivalence, we used the second negation law.

(4) For the fourth equivalence, we used the first identity law.

(5) For the fifth equivalence, we used the first commutative law.

(6) For the sixth equivalence, we used the second commutative law.

(7) For the last equivalence, we used the first absorption law.

Exercise 1.30: Show that the statement $[(\neg p \vee q) \wedge p] \vee q$ is logically equivalent to q.

____

____

____

Tautologies and Contradictions

A statement that has truth value T for all truth assignments of the propositional variables is called a **tautology**. Similarly, a statement that has truth value F for all truth assignments of the propositional variables is called a **contradiction**.

Example 1.31: Let's show that the statement $p \to p$ is a tautology.

Direct method: If $p \equiv \mathrm{T}$, then $p \to p \equiv \mathrm{T} \to \mathrm{T} \equiv \mathrm{T}$. If $p \equiv \mathrm{F}$, then $p \to p \equiv \mathrm{F} \to \mathrm{F} \equiv \mathrm{T}$. Since both possible truth assignments of the propositional variable p lead to the statement $p \to p$ having truth value T, it follows that $p \to p$ is a tautology.

Truth table method:

p	$p \to p$
T	**T**
F	**T**

Since the last column of the truth table consists of only the truth value T, the statement $p \to p$ is a tautology.

Exercise 1.32: Use a truth table to show that $(p \to q) \leftrightarrow (\neg q \to \neg p)$ is a tautology.

Note: Observe the similarity between the tautology $(p \to q) \leftrightarrow (\neg q \to \neg p)$ and the law of the contrapositive $p \to q \equiv \neg q \to \neg p$ (this is logical equivalence 11 from List 1.28). Given any logical equivalence $A \equiv B$ (where A and B are sentences), we always have a corresponding tautology $A \leftrightarrow B$.

Example 1.33: From the first De Morgan's Law $\neg(p \wedge q) \equiv \neg p \vee \neg q$, it follows that the statement $\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$ is a tautology.

Exercise 1.34: Show directly that the statement $p \wedge \neg p$ is a contradiction (by "directly," we mean that you should **not** use a truth table).

Problem Set 1

Full solutions to these problems are available for free download here:

www.SATPrepGet800.com/PMNR2ZX

LEVEL 1

Determine whether each of the following sentences is an atomic statement, a compound statement, or not a statement at all:

1. Grace is not going shopping tomorrow.
2. Where did I go wrong?
3. Stay out of my way today.
4. $x > 26$.
5. I visited the law firm of Cooper and Smith.
6. If there is an elephant in the room, then we need to talk.
7. $2 < -7$ or $15 > 100$.
8. This sentence is six words long.
9. A triangle is equilateral if and only if all three sides of the triangle have the same length.
10. I cannot speak Russian, but I can speak Spanish.

What is the negation of each of the following statements?

11. Cauliflower is Jamie's favorite vegetable.
12. We have three cats.
13. $15 < -12$.
14. You are not serious.
15. The function f is continuous.
16. The real number system with the standard topology is locally compact.

Level 2

Let p represent the statement “5 is an odd integer,” let q represent the statement “Brazil is in Europe,” and let r represent the statement “A lobster is an insect.” Rewrite each of the following symbolic statements in words, and state the truth value of each statement:

17. $p \vee q$

18. $\neg r$

19. $p \to q$

20. $p \leftrightarrow r$

21. $\neg q \wedge r$

22. $\neg(p \wedge q)$

23. $\neg p \vee \neg q$

24. $(p \wedge q) \to r$

Consider the compound sentence “You can have a cookie or ice cream.” In English this would most likely mean that you can have one or the other but not both. The word “or” used here is generally called an “exclusive or” because it excludes the possibility of both. The disjunction is an “inclusive or.”

25. Using the symbol $\oplus$ for exclusive or, draw the truth table for this connective.

26. Using only the logical connectives $\neg$, $\wedge$, and $\vee$, produce a statement using the propositional variables p and q that has the same truth values as $p \oplus q$.

Level 3

Consider the four distinct propositional variables p, q, r, and s.

27. How many different truth assignments are there for this list of propositional variables?

28. How many different truth assignments are there for this list of propositional variables such that p is true and q is false?

29. How many different truth assignments are there for this list of propositional variables such that q, r, and s are all true?

30. How many different truth assignments are there for a list of 5 propositional variables?

Let p, q, and r represent true statements. Compute the truth value of each of the following compound statements:

31. $(p \vee q) \vee r$

32. $(p \vee q) \wedge \neg r$

33. $\neg p \to (q \vee r)$

34. $\neg(p \leftrightarrow \neg q) \wedge r$

35. $\neg[p \wedge (\neg q \to r)]$

36. $\neg[(\neg p \vee \neg q) \leftrightarrow \neg r]$

37. $p \to (q \to \neg r)$

38. $\neg[\neg p \to (q \to \neg r)]$

Determine if each of the following statements is a tautology, a contradiction, or neither.

39. $p \wedge p$

40. $p \wedge \neg p$

41. $(p \vee \neg p) \to (p \wedge \neg p)$

42. $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$

43. $p \to (\neg q \wedge r)$

44. $(p \leftrightarrow q) \to (p \to q)$

LEVEL 4

Assume that the given compound statement is true. Determine the truth value of each propositional variable.

45. $p \wedge q$

46. $\neg(p \to q)$

47. $p \leftrightarrow [\neg(p \wedge q)]$

48. $[p \wedge (q \vee r)] \wedge \neg r$

Let p represent a true statement. Decide if this is enough information to determine the truth value of each of the following statements. If so, state that truth value.

49. $p \vee q$

50. $p \rightarrow q$

51. $\neg p \rightarrow \neg(q \vee \neg r)$

52. $\neg(\neg p \wedge q) \leftrightarrow p$

53. $(p \leftrightarrow q) \leftrightarrow \neg p$

54. $\neg[(\neg p \wedge \neg q) \leftrightarrow \neg r]$

55. $[(p \wedge \neg p) \rightarrow p] \wedge (p \vee \neg p)$

56. $r \rightarrow [\neg q \rightarrow (\neg p \rightarrow \neg r)]$

For each of the following pairs of statements A and B, show that $A \equiv B$.

57. $A = p \wedge q,\, B = q \wedge p$

58. $A = (p \vee q) \vee r,\, B = p \vee (q \vee r)$

59. $A = \ p \wedge (q \vee r),\, B = (p \wedge q) \vee (p \wedge r)$

60. $A = (p \vee q) \wedge p,\, B = p$

61. $A = p \leftrightarrow q,\, B = (p \rightarrow q) \wedge (q \rightarrow p)$

62. $A = \neg(p \rightarrow q),\, B = p \wedge \neg q$

Simplify each statement.

63. $p \vee (p \wedge \neg p)$

64. $(p \wedge q) \vee \neg p$

65. $\neg p \rightarrow (\neg q \rightarrow p)$

66. $(p \wedge \neg q) \vee p$

67. $[(q \wedge p) \vee q] \wedge [(q \vee p) \wedge p]$

LEVEL 5

Without drawing a truth table or using List 1.28, show that each of the following is a tautology.

68. $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee (p \wedge r)]$

69. $\big[[(p \wedge q) \to r] \to s\big] \to [(p \to r) \to s]$

Let n be a positive integer (in other words, n is one of the numbers $1, 2, 3, 4, \ldots$) and let A be a statement involving n propositional variables. Determine how many rows are in the truth table for A if n is equal to each of the following:

70. $n = 6$

71. $n = 10$

72. n is an arbitrary positive integer (provide an explicit expression involving n)

CHALLENGE PROBLEMS

73. Determine a tautology or contradiction containing *at least* three propositional variables and *at least* three logical connectives so that the truth values for all truth assignments can be evaluated with *no more than* 5 computations and such that *at least* 3 computations are required. Write out your compound statement, followed by your 3 to 5 computations.

74. Let T be a truth table. Explain why there is a statement A involving only the logical connectives $\wedge$, $\vee$, and $\neg$ such that the truth table of A is T.

    Hint: For example, let T be the following truth table:

p	q	?
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**F**

If we let A be the statement $(p \wedge q) \vee (\neg p \wedge q)$, then the truth table of A is T.

Solutions to Exercises

Lesson 1

Exercise 1.3:

1. This is a statement. Either the check is in the mail (in which case the statement is true) or the check is not in the mail (in which case the statement is false).
2. This is **not** a statement. It is a **question**.
3. This is a statement. This statement happens to be false.
4. This is **not** a statement. It is a **command**. This particular command is an **idiom**. This means that the actual meaning of the sentence cannot be deduced from the individual words in the sentence. In this particular case, the meaning of the sentence is that you shouldn't accept something as determined before it has actually occurred. Notice how the actual meaning of the sentence does not involve chickens or hatching.
5. This is a statement. Either Odin is chasing a mouse or he is not. **Side note:** in case you haven't figured it out from the context of the statement, Odin is a cat.

Exercise 1.6:

1. This is an **atomic statement**.
2. This is a **compound statement**. It uses the logical connective "or."
3. This is a **compound statement**. It uses the logical connective "not."
4. This is an **atomic statement**. Even though the word "and" appears in the statement, here it is part of the name of the show. It is not being used as a logical connective.
5. This is a **compound statement**. It uses the logical connective "if…then."
6. This is an **atomic statement**.
7. This is a **compound** statement. It actually uses two connectives: "or" and "not."
8. This is a **compound statement**. It uses the logical connective "if and only if."
9. This is an **atomic statement**. Even though the words "and" and "or" appear in the statement, they are **not** being used as logical connectives.
10. This is a **compound statement**. Like part 7 above, it uses two connectives: "and" and "not." Note that in sentential logic the word "but" has the same meaning as the word "and." In English, the word "but" is used to introduce contrast with the part of the sentence that has already been mentioned. However, logically it is no different from "and."

Exercise 1.10: There are **eight** possible truth assignments for this list of propositional variables. We can visualize this list of truth assignments with the following table:

p	q	r
T	**T**	**T**
T	**T**	**F**
T	**F**	**T**
T	**F**	**F**
F	**T**	**T**
F	**T**	**F**
F	**F**	**T**
F	**F**	**F**

Exercise 1.12:

1. $p \wedge q \equiv \mathrm{T} \wedge \mathrm{T} \equiv \mathbf{T}$.
2. $p \wedge q \equiv \mathrm{F} \wedge \mathrm{F} \equiv \mathbf{F}$.
3. $p \wedge q \equiv \mathrm{F} \wedge \mathrm{T} \equiv \mathbf{F}$.

p	q	$p \wedge q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

Exercise 1.13:

1. $p \vee q \equiv \mathrm{T} \vee \mathrm{T} \equiv \mathbf{T}$.
2. $p \vee q \equiv \mathrm{F} \vee \mathrm{F} \equiv \mathbf{F}$.
3. $p \vee q \equiv \mathrm{T} \vee \mathrm{F} \equiv \mathbf{T}$.
4. $p \vee q \equiv \mathrm{F} \vee \mathrm{T} \equiv \mathbf{T}$.

p	q	$p \vee q$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

Exercise 1.14:

1. $p \rightarrow q \equiv \mathrm{T} \rightarrow \mathrm{T} \equiv \mathbf{T}$.
2. $p \rightarrow q \equiv \mathrm{F} \rightarrow \mathrm{F} \equiv \mathbf{T}$.
3. $p \rightarrow q \equiv \mathrm{T} \rightarrow \mathrm{F} \equiv \mathbf{F}$.
4. $p \rightarrow q \equiv \mathrm{F} \rightarrow \mathrm{T} \equiv \mathbf{T}$.

p	q	$p \rightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Exercise 1.15:

1. $p \leftrightarrow q \equiv \mathrm{T} \leftrightarrow \mathrm{T} \equiv \mathbf{T}$.
2. $p \leftrightarrow q \equiv \mathrm{F} \leftrightarrow \mathrm{F} \equiv \mathbf{T}$.
3. $p \leftrightarrow q \equiv \mathrm{T} \leftrightarrow \mathrm{F} \equiv \mathbf{F}$.
4. $p \leftrightarrow q \equiv \mathrm{F} \leftrightarrow \mathrm{T} \equiv \mathbf{F}$.

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

p	$\neg p$
T	**F**
F	**T**

Exercise 1.17: $\neg p \equiv \neg \mathrm{F} \equiv \mathbf{T}$.

Exercise 1.19: Note that p and q are both false.

1. $p \to q$ represents **"If frogs are birds, then $2 < 1$."** Since p is false, $p \to q$ is **true**.
2. $\neg p \vee q$ represents the statement **"Frogs are not birds or $2 < 1$."** Since $\neg p$ is true, $\neg p \vee q$ is **true**. Note once again that $\neg p \vee q$ always means $(\neg p) \vee q$. In general, without parentheses present, we always apply negation before any of the other connectives.
3. $p \leftrightarrow q$ represents **"Frogs are birds if and only if $2 < 1$."** Since p and q are both false, $p \leftrightarrow q$ is **true**.
4. $(p \to q) \wedge (q \to p)$ represents **"If frogs are birds, then $2 < 1$ and if $2 < 1$, then frogs are birds."** Since p is false, $p \to q$ is true. Since q is false, $q \to p$ is true. Since $p \to q$ and $q \to p$ are both true, $(p \to q) \wedge (q \to p)$ is **true**.
5. $\neg(p \wedge q)$ represents the statement **"It is not the case that both frogs are birds and $2 < 1$."** Since p and q are both false, $p \wedge q$ is false. It follows that $\neg(p \wedge q)$ is **true**.
6. $\neg p \vee \neg q$ represents the statement **"Frogs are not birds or 2 is not less than 1."** Since p and q are both false, $\neg p$ and $\neg q$ are both true. It follows that $\neg p \vee \neg q$ is **true**.

Exercise 1.21:

1.

p	q	r	$\neg r$	$q \wedge \neg r$	$p \leftrightarrow (q \wedge \neg r)$
T	**T**	**T**	**F**	**F**	**F**
T	**T**	**F**	**T**	**T**	**T**
T	**F**	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**F**	**F**
F	**T**	**T**	**F**	**F**	**T**
F	**T**	**F**	**T**	**T**	**F**
F	**F**	**T**	**F**	**F**	**T**
F	**F**	**F**	**T**	**F**	**T**

2. We use the highlighted row in the truth table above to get a truth value of **false**.
3. $p \leftrightarrow (q \wedge \neg r) \equiv \mathrm{T} \leftrightarrow (q \wedge \neg \mathrm{T}) \equiv \mathrm{T} \leftrightarrow (q \wedge \mathrm{F}) \equiv \mathrm{T} \leftrightarrow \mathrm{F} \equiv \mathrm{F}$. So, the truth value **can be determined** and it is **false**.

Exercise 1.23: We will show that the truth tables for $p \to q$ and $\neg p \vee q$ are the same.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Exercise 1.25: We will show that the truth tables for $p \to q$ and $\neg q \to \neg p$ are the same.

p	q	$p \to q$	$\neg p$	$\neg q$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

Exercise 1.27: We will show that the truth tables for $\neg(p \vee q)$and $\neg p \wedge \neg q$ are the same.

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Exercise 1.30:

$$[(\neg p \vee q) \wedge p] \vee q \equiv [p \wedge (\neg p \vee q)] \vee q \equiv [(p \wedge \neg p) \vee (p \wedge q)] \vee q \equiv [\mathrm{F} \vee (p \wedge q)] \vee q$$
$$\equiv [(p \wedge q) \vee \mathrm{F}] \vee q \equiv (p \wedge q) \vee q \equiv (q \wedge p) \vee q \equiv q$$

So, we see that $[(\neg p \vee q) \wedge p] \vee q$ is logically equivalent to the atomic statement q.

Notes: (1) For the first equivalence, we used the first commutative law.

(2) For the second equivalence, we used the first distributive law.

(3) For the third equivalence, we used the first negation law.

(4) For the fourth equivalence, we used the second commutative law.

(5) For the fifth equivalence, we used the fourth identity law.

(6) For the sixth equivalence, we used the first commutative law.

(7) For the last equivalence, we used the second absorption law.

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About the Author

Dr. Steve Warner, a New York native, earned his Ph.D. at Rutgers University in Pure Mathematics in May 2001. While a graduate student, Dr. Warner won the TA Teaching Excellence Award.

After Rutgers, Dr. Warner joined the Penn State Mathematics Department as an Assistant Professor and in September 2002, he returned to New York to accept an Assistant Professor position at Hofstra University. By September 2007, Dr. Warner had received tenure and was promoted to Associate Professor. He has taught undergraduate and graduate courses in Precalculus, Calculus, Linear Algebra, Differential Equations, Mathematical Logic, Set Theory, and Abstract Algebra.

From 2003 – 2008, Dr. Warner participated in a five-year NSF grant, "The MSTP Project," to study and improve mathematics and science curriculum in poorly performing junior high schools. He also published several articles in scholarly journals, specifically on Mathematical Logic.

Dr. Warner has nearly two decades of experience in general math tutoring and tutoring for standardized tests such as the SAT, ACT, GRE, GMAT, and AP Calculus exams. He has tutored students both individually and in group settings.

In February 2010 Dr. Warner released his first SAT prep book "The 32 Most Effective SAT Math Strategies," and in 2012 founded Get 800 Test Prep. Since then Dr. Warner has written books for the SAT, ACT, SAT Math Subject Tests, AP Calculus exams, and GRE. In 2018 Dr. Warner released his first pure math book called "Pure Mathematics for Beginners." Since then he has released several more books, each one addressing a specific subject in pure mathematics.

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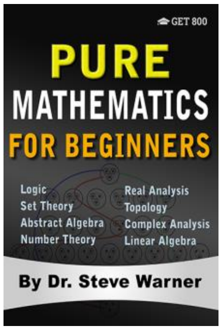

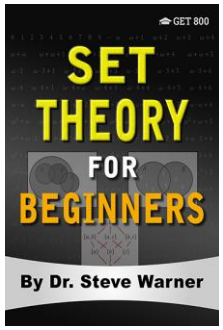

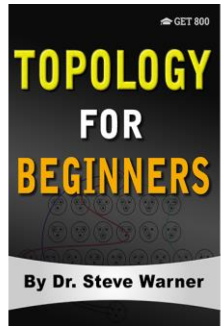

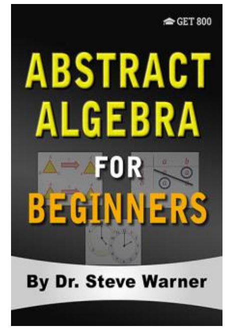

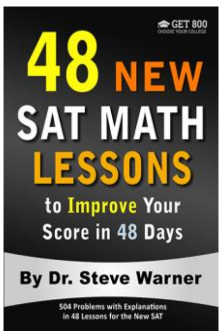

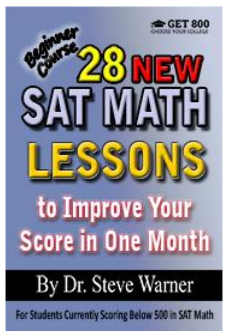

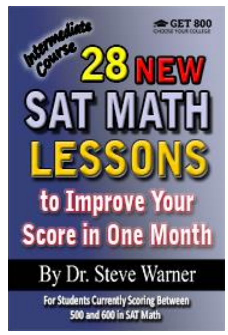

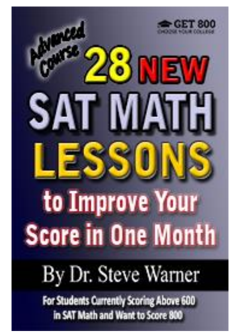

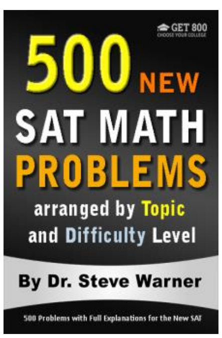

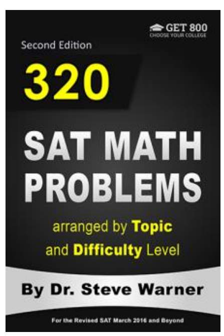

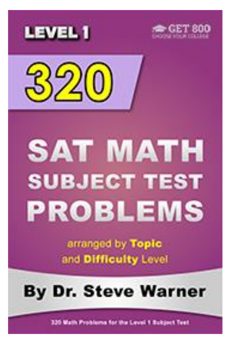

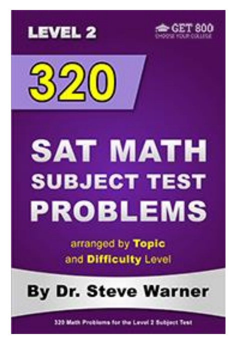

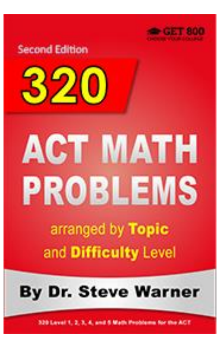

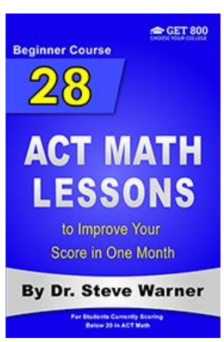

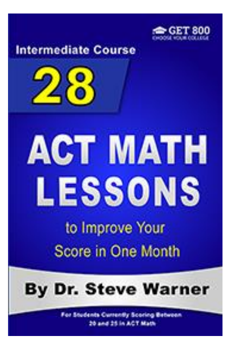

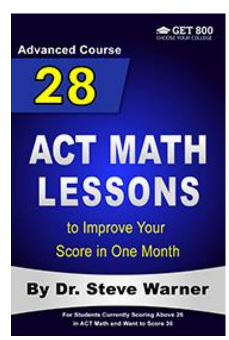

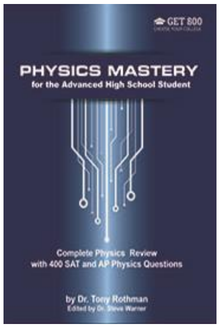

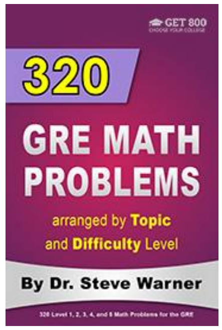

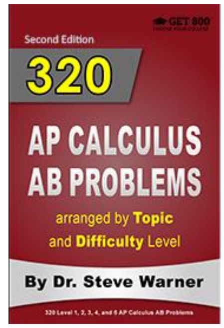

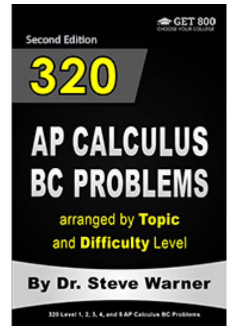

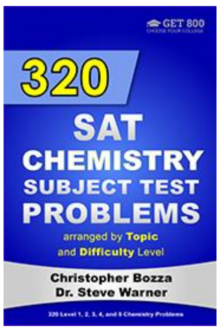

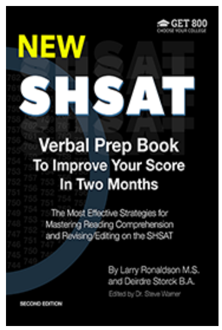

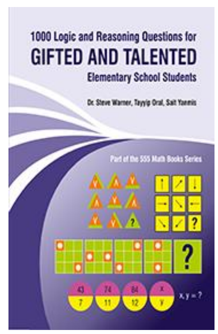

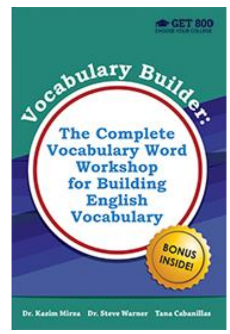

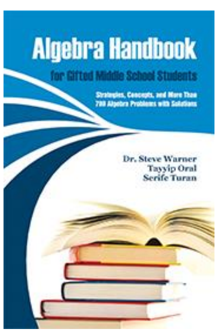

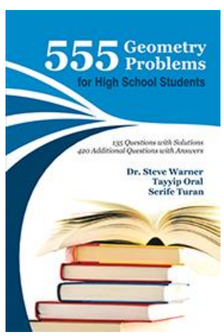

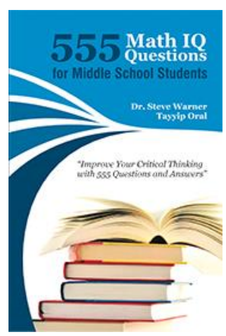

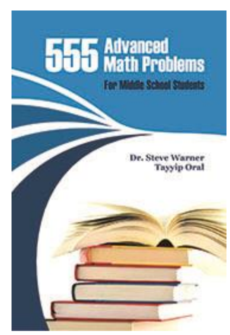

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