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arranged by **Topic** and **Difficulty** Level

By Dr. Steve Warner

320 Level 1, 2, 3, 4, and 5 AP Calculus AB Problems

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Dr. Steve Warner

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Problems by Level and Topic with Fully Explained Solutions

Level 1: Precalculus

1. Let $f(x) = 3$ and $g(x) = x^4 - 2x^3 + x^2 - 5x + 1$. Then $(f \circ g)(x) =$

   (A) 3
   (B) 22
   (C) 10,648
   (D) $3x^4 - 6x^3 + 3x^2 - 15x + 3$

Solution: $(f \circ g)(x) = f\big(g(x)\big) = 3$, choice (A).

Notes: (1) $f(x) = 3$ can be read as "f of *anything* is 3." So, in particular, "f of $g(x)$ is 3," i.e. $f\big(g(x)\big) = 3$.

(2) For completeness we have

$$(f \circ g)(x) = f\big(g(x)\big) = f(x^4 - 2x^3 + x^2 - 5x + 1) = 3.$$

(3) $(f \circ g)(x) = f\big(g(x)\big)$ is called the **composition** of the functions f and g. We literally plug the function g in for x inside the function f. As an additional simple example, if $f(x) = x^2$ and $g(x) = x + 1$, then

$$f\big(g(x)\big) = f(x+1) = (x+1)^2 \quad \text{and} \quad g\big(f(x)\big) = g(x^2) = x^2 + 1$$

Notice how, for example, $f\left(\boxed{\text{something}}\right) = \left(\boxed{\text{something}}\right)^2$.

The "something" here is $x + 1$. Note how we keep it in parentheses.

2. What is the domain of $k(x) = \sqrt[3]{8 - 12x^2 + 6x - x^3}$?

   (A) All real numbers
   (B) $x < -2$
   (C) $-2 < x < 2$
   (D) $x > 2$

Solution: The domain of $f(x) = 8 - 12x^2 + 6x - x^3$ is all real numbers since $f(x)$ is a polynomial. The domain of $g(x) = \sqrt[3]{x}$ is also all real numbers. The function $k(x)$ is the composition of these two functions and therefore also has domain all real numbers. So the answer is choice (A).

Notes: (1) Polynomials and cube roots do not cause any problems. In other words, you can evaluate a polynomial at any real number and you can take the cube root of any real number.

(2) Square roots on the other hand do have some problems. We cannot take the square root of negative real numbers (in the reals). We have the same problem for any even root, and there are no problems for odd roots.

For example $\sqrt{-8}$ is undefined, whereas $\sqrt[3]{-8} = -2$ because

$$(-2)^3 = (-2)(-2)(-2) = -8.$$

(3) See problem 5 for more information about polynomials.

3. If $K(x) = \log_5 x$ for $x > 0$, then $K^{-1}(x) =$

   (A) $\log_x 5$
   (B) 5^x
   (C) $\frac{x}{5}$
   (D) $\frac{5}{x}$

Solution: The inverse of the logarithmic function $K(x) = \log_5 x$ is the exponential function $K^{-1}(x) = 5^x$, choice (B).

Notes: (1) The word “logarithm” just means “exponent.”

(2) The equation $y = \log_b x$ can be read as “y is the exponent when we rewrite x with a base of b.” In other words we are raising b to the power y. So the equation can be written in exponential form as $x = b^y$.

In this problem $b = 5$, and so the logarithmic equation $y = \log_5 x$ can be written in exponential form as $x = 5^y$.

(3) In general, the functions $y = b^x$ and $y = \log_b x$ are inverses of each other. In fact, that is precisely the definition of a logarithm.

(4) The usual procedure to find the inverse of a function $y = f(x)$ is to interchange the roles of x and y and solve for y. In this example, the inverse of $y = \log_5 x$ is $x = \log_5 y$. To solve this equation for y we can simply change the equation to its exponential form $y = 5^x$.

4. If $g(x) = e^{x+1}$, which of the following lines is an asymptote to the graph of $g(x)$?

(A) $x = 0$
(B) $y = 0$
(C) $x = -1$
(D) $y = -1$

Solution: The graph of $y = e^x$ has a horizontal asymptote of $y = 0$. To get the graph of $y = e^{x+1}$ we shift the graph of $y = e^x$ to the left one unit. A horizontal shift does not have any effect on a horizontal asymptote. So the answer is $y = 0$, choice (B).

Notes: (1) It is worth reviewing the following basic transformations:

Let $y = f(x)$, and $k > 0$. We can move the graph of f around by applying the following basic transformations.

$y = f(x) + k$	shift up k units
$y = f(x) - k$	shift down k units
$y = f(x - k)$	shift right k units
$y = f(x + k)$	shift left k units
$y = -f(x)$	reflect in x-axis
$y = f(-x)$	reflect in y-axis.

For the function $g(x) = e^{x+1}$, we are replacing x by $x + 1$ in $f(x) = e^x$. In other words, $g(x) = f(x + 1)$. So we get the graph of g by shifting the graph of f 1 unit to the left.

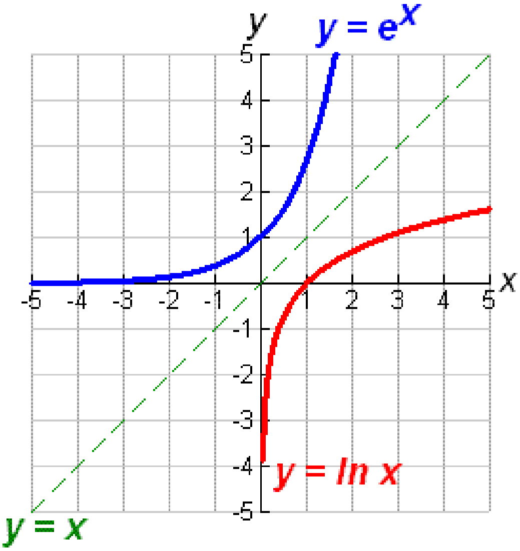
(2) The horizontal line with equation $y = b$ is a **horizontal asymptote** for the graph of the function $y = f(x)$ if y approaches b as x gets larger and larger, or smaller and smaller (as in very large in the negative direction).

(3) We can also find a horizontal asymptote by plugging into our calculator a really large negative value for x such as $-999{,}999{,}999$ (if a calculator is allowed for the problem). We get $e^{-999{,}999{,}999+1} = 0$.

(Note that the answer is not really zero, but the calculator gives an answer of 0 because the actual answer is so close to 0 that the calculator cannot tell the difference.)

(4) It is worth memorizing that $\boldsymbol{y = e^x}$ **has a horizontal asymptote of** $\boldsymbol{y = 0}$**, and** $\boldsymbol{y = \ln x}$ **has a vertical asymptote of** $\boldsymbol{x = 0}$.

As an even better alternative, you should be able to visualize the graphs of both of these functions. Here is a picture.



5. Which of the following equations has a graph that is symmetric with respect to the y-axis?

(A) $y = (x+1)^3 - x$
(B) $y = (x+1)^2 - 1$
(C) $y = 2x^6 - 3x^2 + 5$
(D) $y = \frac{x-1}{2x}$

Solution: The equation $y = 2x^6 - 3x^2 + 5$ is a polynomial equation with only even exponents. It is therefore an even function, and so its graph is symmetric with respect to the y-axis. So the answer is (C).

Notes: (1) A function f with the property that $f(-x) = f(x)$ for all x is called an **even** function. Even functions have graphs which are **symmetric with respect to the y-axis**.

(2) A function f with the property that $f(-x) = -f(x)$ for all x is called an **odd** function. Odd functions have graphs which are **symmetric with respect to the origin**.

(3) A **polynomial** has the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where a_0, a_1,…,a_n are real numbers. For example, $2x^6 - 3x^2 + 5$ is a polynomial.

(4) Polynomial functions with only even powers of x are even functions (and therefore are symmetric with respect to the y-axis). Keep in mind that a constant c is the same as cx^0 , and so c is an even power of x. For example 5 is an even power of x. From this observation we can see that the polynomial in answer choice (C) is an even function.

(5) Polynomial functions with only odd powers of x are odd functions (and therefore are symmetric with respect to the origin). Keep in mind that x is the same as x^1, and so x is an odd power of x. An example of an odd function is the polynomial $y = -4x^3 + 2x$.

(6) Note that the functions given in answer choices (A) and (B) are also polynomials. This can be seen by expanding the given expressions. For example, let's look at the function given in choice (B).

$$(x+1)^2 - 1 = (x+1)(x+1) - 1 = x^2 + x + x + 1 - 1 = x^2 + 2x.$$

Since this expression has both an even and an odd power of x, the polynomial given in choice (B) is neither even nor odd.

I leave it as an exercise to expand $(x+1)^3 - x$ and observe that it has both even and odd powers of x.

(7) A **rational function** is a quotient of polynomials. The function given in choice (D) is a rational function. To determine if this function is even we need to use the definition of being even:

$$\frac{(-x)-1}{2(-x)} = \frac{-x-1}{-2x} = \frac{x+1}{2x} \neq \frac{x-1}{2x}$$

It follows that the function is not even, and therefore its graph is not symmetric with respect to the y-axis.

(8) The graph of an even function is **symmetric with respect to the y-axis**. This means that the y-axis acts like a "mirror," and the graph "reflects" across this mirror. If you put choice (C) into your graphing calculator, you will see that this graph has this property.

Similarly, the graph of an odd function is **symmetric with respect to the origin**. This means that if you rotate the graph 180 degrees (or equivalently, turn it upside down) it will look the same as it did right side up. If you put the polynomial $y = -4x^3 + 2x$ into your graphing calculator, you will see that this graph has this property.

Put the other three answer choices in your graphing calculator and observe that they have neither of these symmetries.

(9) If we were allowed to use a calculator, we could solve this problem by graphing each equation, and checking to see if the y-axis acts like a mirror.

6. If $f(x) = x^3 + Ax^2 + Bx + C$, and if $f(0) = -2$, $f(-1) = 7$, and $f(1) = 4$, then $AB + \frac{3}{4} =$

   (A) 36
   (B) 18
   (C) -18
   (D) It cannot be determined from the information given

Solution: Since $f(0) = -2$, we have

$$-2 = 0^3 + A(0)^2 + B(0) + C = C.$$

Since $f(-1) = 7$, we have

$$7 = (-1)^3 + A(-1)^2 + B(-1) + C = -1 + A - B + C.$$

We already found that $C = -2$, so we have

$$7 = -1 + A - B - 2 = A - B - 3.$$

So $A - B = 10$.

Since $f(1) = 4$, we have

$$4 = 1^3 + A(1)^2 + B(1) + C = 1 + A + B - 2 = A + B - 1.$$

So $A + B = 5$.

Let's add these last two equations.

$$\begin{array}{l} A - B = 10 \\ \underline{A + B = \ \ 5} \\ 2A \qquad = 15 \end{array}$$

It follows that $A = \frac{15}{2}$, and so $B = 5 - A = 5 - \frac{15}{2} = -\frac{5}{2}$.

Finally, $AB + \frac{3}{4} = \left(\frac{15}{2}\right)\left(-\frac{5}{2}\right) + \frac{3}{4} = -\frac{75}{4} + \frac{3}{4} = -\frac{72}{4} = -18$, choice (C).

7. If the solutions of $g(x) = 0$ are $-3, \frac{1}{2}$ and 5, then the solutions of $g(3x) = 0$ are

(A) -1, $\frac{1}{6}$ and $\frac{5}{3}$
(B) -9, $\frac{3}{2}$ and 15
(C) -6, $-\frac{5}{2}$ and 2
(D) 0, $\frac{7}{2}$ and 8

Solution: We simply set $3x$ equal to -3, $\frac{1}{2}$, and 5, and then solve each of these equations for x. We get -1, $\frac{1}{6}$, and $\frac{5}{3}$. So the answer is choice (A).

Notes: (1) Since -3 is a solution of $g(x) = 0$, it follows that $g(-3) = 0$. So we have $g\big(3(-1)\big) = 0$. So -1 is a solution of $g(3x) = 0$.

(2) To get that -1 is a solution of $g(3x) = 0$ formally, we simply divide -3 by 3, or equivalently, we solve the equation $3x = -3$ for x.

(3) Similarly, we have $g\left(3\left(\frac{1}{6}\right)\right) = 0$ and $g\left(3\left(\frac{5}{3}\right)\right) = 0$, and we can formally find that $\frac{1}{6}$ and $\frac{5}{3}$ are solutions of $g(3x) = 0$ by solving the equations $3x = \frac{1}{2}$ and $3x = 5$.

8. If $k(x) = \frac{x^2-1}{x+2}$ and $h(x) = \ln x^2$, then $k\big(h(e)\big) =$

(A) 0.12
(B) 0.50
(C) 0.51
(D) 0.75

Solution: $h(e) = \ln e^2 = 2$. So $k(h(e)) = k(2) = \frac{2^2-1}{2+2} = 0.75$, choice (D).

Notes: (1) We first substituted e into the function h to get 2. We then substituted 2 into the function k to get 0.75.

(2) See the notes at the end of problem 3 for more information on logarithms.

(3) There are several ways to compute $\ln e^2$.

Method 1: Simply use your calculator (if allowed).

Method 2: Recall that the functions e^x and $\ln x$ are inverses of each other. This means that $e^{\ln x} = x$ and $\ln e^x = x$. Substituting $x = 2$ into the second equation gives the desired result.

Method 3: Remember that $\ln x = \log_e x$. So we can rewrite the equation $y = \ln e^2$ in exponential form as $e^y = e^2$. So $y = 2$.

Method 4: Recall that $\ln e = 1$. We have $\ln e^2 = 2 \ln e = 2(1) = 2$. Here we have used the last law in the following table:

Laws of Logarithms: Here is a review of the basic laws of logarithms.

Law	Example
$\log_b 1 = 0$	$\log_2 1 = 0$
$\log_b b = 1$	$\log_6 6 = 1$
$\log_b x + \log_b y = \log_b(xy)$	$\log_5 7 + \log_5 2 = \log_5 14$
$\log_b x - \log_b y = \log_b(\frac{x}{y})$	$\log_3 21 - \log_3 7 = \log_3 3 = 1$
$\log_b x^n = n\log_b x$	$\log_8 3^5 = 5\log_8 3$

LEVEL 1: DIFFERENTIATION

9. If $f(x) = x^2 + x - \cos x$, then $f'(x) =$

(A) $2x + 1 - \sin x$
(B) $2x + 1 + \sin x$
(C) $2x - \sin x$
(D) $\frac{1}{3}x^3 + \frac{1}{2}x^2 - \sin x$

Solution: $f'(x) = 2x + 1 + \sin x$. This is choice (B).

Notes: (1) If n is any real number, then the derivative of x^n is nx^{n-1}.

Symbolically, $\boldsymbol{\frac{d}{dx}[x^n] = nx^{n-1}}$.

For example, $\frac{d}{dx}[x^2] = 2x^1 = 2x$.

As another example, $\frac{d}{dx}[x] = \frac{d}{dx}[x^1] = 1x^0 = 1(1) = 1$.

(2) Of course it is worth just remembering that $\frac{d}{dx}[x] = 1$.

(3) You should know the derivatives of the six basic trig functions:

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

(4) If g and h are functions, then $(g+h)'(x) = g'(x) + h'(x)$.

In other words, when differentiating a sum, we can simply differentiate term by term.

Similarly, $(g-h)'(x) = g'(x) - h'(x)$.

(5) In the given problem we differentiate each of x^2, x, and $\cos x$ separately and then use note (4) to write the final answer.

10. If $g(x) = \frac{x+2}{x-2}$, then $g'(-2) =$

(A) $-\frac{1}{4}$

(B) -1

(C) 1

(D) $\frac{1}{4}$

Solution: $g'(x) = \frac{(x-2)(1)-(x+2)(1)}{(x-2)^2} = \frac{x-2-x-2}{(x-2)^2} = \frac{-4}{(x-2)^2}$. So we have $g'(-2) = \frac{-4}{(-2-2)^2} = \frac{-4}{(-4)^2} = \frac{-4}{16} = -\frac{1}{4}$, choice (A).

Notes: (1) We used the **quotient rule** which says the following:

If $f(x) = \frac{N(x)}{D(x)}$, then

$$f'(x) = \frac{D(x)N'(x) - N(x)D'(x)}{[D(x)]^2}$$

I like to use the letters N for "numerator" and D for "denominator."

(2) The derivative of $x+2$ is 1 because the derivative of x is 1, and the derivative of any constant is 0.

Similarly, the derivative of $x-2$ is also 1.

(3) If we could use a calculator for this problem, we can compute $g'(-2)$ using our TI-84 calculator by first selecting nDeriv((or pressing 8) under the MATH menu, then typing the following: (X + 2)/(X − 2), X, −2), and pressing ENTER. The display will show approximately $-.25$.

11. If $h(x) = \frac{1}{12}x^3 - 2\ln x + \sqrt{x}$, then $h'(x) =$

(A) $\frac{1}{3}x^2 - \frac{2}{x} + \frac{1}{2\sqrt{x}}$

(B) $\frac{1}{4}x^2 - \frac{2}{\ln x} + \frac{2}{3}x^{\frac{3}{2}}$

(C) $\frac{1}{4}x^2 - \frac{2}{x} + \frac{1}{2\sqrt{x}}$

(D) $\frac{1}{4}x^2 - \frac{2}{\ln x} + \frac{1}{2\sqrt{x}}$

Solution: $h'(x) = \frac{1}{12} \cdot 3x^2 - 2\left(\frac{1}{x}\right) + \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{4}x^2 - \frac{2}{x} + \frac{1}{2\sqrt{x}}$, choice (C).

Notes: (1) The derivative of a constant times a function is the constant times the derivative of the function.

Symbolically, $\frac{d}{dx}[cg(x)] = c\frac{d}{dx}[g(x)]$.

For example, $\frac{d}{dx}\left[\frac{1}{12}x^3\right] = \frac{1}{12}\frac{d}{dx}[x^3] = \frac{1}{12}(3x^2) = \frac{1}{4}x^2$.

(2) The derivative of $\ln x$ is $\frac{1}{x}$.

Symbolically, $\frac{d}{dx}[\ln x] = \frac{1}{x}$.

(3) Combining (1) and (2), we have $\frac{d}{dx}[2\ln x] = 2\frac{d}{dx}[\ln x] = 2\left(\frac{1}{x}\right) = \frac{2}{x}$.

(4) $\sqrt{x}$ can be written as $x^{\frac{1}{2}}$. So the derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-\frac{1}{2}}$.

(5) $x^{-\frac{1}{2}} = \frac{1}{x^{\frac{1}{2}}} = \frac{1}{\sqrt{x}}$.

(6) Combining (4) and (5) we have $\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

(7) Here is a review of the laws of exponents:

Law	Example
$x^0 = 1$	$3^0 = 1$
$x^1 = x$	$9^1 = 9$
$x^a x^b = x^{a+b}$	$x^3 x^5 = x^8$
$x^a/x^b = x^{a-b}$	$x^{11}/x^4 = x^7$
$(x^a)^b = x^{ab}$	$(x^5)^3 = x^{15}$
$(xy)^a = x^a y^a$	$(xy)^4 = x^4 y^4$
$(x/y)^a = x^a/y^a$	$(x/y)^6 = x^6/y^6$
$x^{-1} = 1/x$	$3^{-1} = 1/3$
$x^{-a} = 1/x^a$	$9^{-2} = 1/81$
$x^{1/n} = \sqrt[n]{x}$	$x^{1/3} = \sqrt[3]{x}$
$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$	$x^{9/2} = \sqrt{x^9} = \left(\sqrt{x}\right)^9$

12. The slope of the tangent line to the graph of $y = e^{3x}$ at $x = \ln 2$ is

(A) $8 \ln 2$
(B) 8
(C) $24 \ln 2$
(D) 24

Solution: $y' = e^{3x}(3) = 3e^{3x}$. When $x = \ln 2$, we have that the slope of the tangent line is

$$y'|_{x=\ln 2} = 3e^{3\ln 2} = 3e^{\ln 2^3} = 3e^{\ln 8} = 3(8) = 24.$$

This is choice (D).

Notes: (1) To find the slope of a tangent line to the graph of a function, we simply take the derivative of that function. If we want the slope of the tangent line at a specified x-value, we substitute that x-value into the derivative of the function.

(2) The derivative of $f(x) = e^x$ is $f'(x) = e^x$

(3) In this problem we used the **chain rule** which says the following:

If $f(x) = (g \circ h)(x) = g(h(x))$, then

$$f'(x) = g'\big(h(x)\big) \cdot h'(x)$$

Here we have $h(x) = 3x$ and $g(x) = e^x$. So $h'(x) = 3$, and

$$g'(h(x)) = e^{h(x)} = e^{3x}.$$

(4) See the notes at the end of problem 3 for information on logarithms.

(5) $\ln x$ is an abbreviation for $\log_e x$.

(6) The functions e^x and $\ln x$ are inverses of each other. This means that $e^{\ln x} = x$ and $\ln e^x = x$.

(7) $n \ln x = \ln x^n$

(8) Using notes (6) and (7) together we get $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$.

(9) See the notes at the end of problem 8 for a review of the laws of logarithms.

(10) If we could use a calculator for this problem, we can compute y' at $x = \ln 2$ using our TI-84 calculator by first selecting nDeriv((or pressing 8) under the MATH menu, then typing the following: e^(3X), X, ln 2), and pressing ENTER. The display will show approximately 24.

13. The instantaneous rate of change at $x = 3$ of the function $f(x) = x\sqrt{x+1}$ is

    (A) $\frac{1}{4}$

    (B) $\frac{3}{4}$

    (C) $\frac{5}{4}$

    (D) $\frac{11}{4}$

Solution: $f'(x) = x \cdot \frac{1}{2}(x+1)^{-\frac{1}{2}} + \sqrt{x+1}(1) = \frac{x}{2\sqrt{x+1}} + \sqrt{x+1}$.

So $f'(3) = \frac{3}{2\sqrt{3+1}} + \sqrt{3+1} = \frac{3}{2\sqrt{4}} + \sqrt{4} = \frac{3}{2 \cdot 2} + 2 = \frac{3}{4} + \frac{8}{4} = \frac{11}{4}$, choice (D).

Notes: (1) We used the **product rule** which says the following:

If $f(x) = u(x)v(x)$, then

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$

(2) The derivative of x is 1.

(3) $\sqrt{x+1}$ can be written as $(x+1)^{\frac{1}{2}}$. So the derivative of $\sqrt{x+1}$ is $\frac{1}{2}(x+1)^{-\frac{1}{2}}$ (technically we need to use the chain rule here and also take the derivative of $x+1$, but $\frac{d}{dx}[x+1]$ is just 1).

(4) $(x+1)^{-\frac{1}{2}} = \frac{1}{(x+1)^{\frac{1}{2}}} = \frac{1}{\sqrt{x+1}}$.

(5) Combining (3) and (4) we have

$$\frac{d}{dx}\left[\sqrt{x+1}\right] = \frac{d}{dx}\left[(x+1)^{\frac{1}{2}}\right] = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}}.$$

(6) See problem 11 for a review of all of the laws of exponents you should know.

(7) If we can use a calculator for this problem, we can compute $f'(3)$ using our TI-84 calculator by first selecting nDeriv((or pressing 8) under the MATH menu, then typing the following: X√(X+1), X, 3), and pressing ENTER. The display will show approximately 2.75. Type 2.75, then press MATH ENTER ENTER to change this decimal to $\frac{11}{4}$.

14. $\frac{d}{dx}\left[e^5 + \frac{1}{\sqrt[3]{x^2}} + 11^x\right] =$

Solution:

$$\frac{d}{dx}\left[e^5 + \frac{1}{\sqrt[3]{x^2}} + 11^x\right] = 0 - \frac{2}{3}x^{-\frac{5}{3}} + 11^x(\ln 11) = -\frac{2}{3\sqrt[3]{x^5}} + (\ln 11)11^x.$$

Notes: (1) e^5 is a constant. Therefore $\frac{d}{dx}[e^5] = 0$.

(2) $\frac{1}{\sqrt[3]{x^2}} = \frac{1}{x^{\frac{2}{3}}} = x^{-\frac{2}{3}}$. So $\frac{d}{dx}\left[\frac{1}{\sqrt[3]{x^2}}\right] = \frac{d}{dx}\left[x^{-\frac{2}{3}}\right] = -\frac{2}{3}x^{-\frac{2}{3}-1} = -\frac{2}{3}x^{-\frac{5}{3}}$.

(3) $-\frac{2}{3}x^{-\frac{5}{3}} = -\frac{2}{3x^{\frac{5}{3}}} = -\frac{2}{3\sqrt[3]{x^5}}$.

(4) If $b > 0$, then $\frac{d}{dx}[b^x] = b^x(\ln b)$.

In particular, $\frac{d}{dx}[11^x] = 11^x(\ln 11)$.

(5) For $b > 0$, $b^x = e^{x \ln b}$.

To see this, first observe that $e^{x \ln b} = e^{\ln b^x}$ by the power rule for logarithms (see problem 8 for the laws of logarithms).

Second, recall that the functions e^x and $\ln x$ are inverses of each other. This means that $e^{\ln x} = x$ and $\ln e^x = x$. Replacing x by b^x in the first formula yields $e^{\ln b^x} = b^x$.

(6) The formula in note (5) gives an alternate method for differentiating 11^x. We can rewrite 11^x as $e^{x \ln 11}$ and use the chain rule. Here are the details:

$$\frac{d}{dx}[11^x] = \frac{d}{dx}\left[e^{x \ln 11}\right] = e^{x \ln 11}(\ln 11) = 11^x(\ln 11).$$

Note that in the last step we rewrote $e^{x \ln 11}$ as 11^x.

(7) There is one more method we can use to differentiate 11^x. We can use **logarithmic differentiation**.

We start by writing $y = 11^x$.

We then take the natural log of each side of this equation: $\ln y = \ln 11^x$.

We now use the power rule for logarithms to bring the x out of the exponent: $\ln y = x \ln 11$.

Now we differentiate implicitly to get $\frac{1}{y} \cdot \frac{dy}{dx} = \ln 11$.

Solve for $\frac{dy}{dx}$ by multiplying each side of the last equation by y to get $\frac{dy}{dx} = y \ln 11$.

Finally, replacing y by 11^x gives us $\frac{dy}{dx} = 11^x(\ln 11)$.

(8) **Logarithmic differentiation** is a general method that can often be used to handle expressions that have exponents with variables.

(9) See problem 45 for more information on implicit differentiation.

15. If $y = x^{\cos x}$, then $y' =$

Solution: We take the natural logarithm of each side of the given equation to get $\ln y = \ln x^{\cos x} = (\cos x)(\ln x)$.

We now differentiate implicitly to get

$$\frac{1}{y}y' = (\cos x)\left(\frac{1}{x}\right) + (\ln x)(-\sin x) = \frac{\cos x - x\,(\ln x)(\sin x)}{x}$$

Multiplying each side of this last equation by y yields

$$y' = y[\frac{\cos x - x\,(\ln x)(\sin x)}{x}] = x^{\cos x}[\frac{\cos x - x\,(\ln x)(\sin x)}{x}]$$

Notes: (1) Since the exponent of the expression $x^{\cos x}$ contains the variable x, we used logarithmic differentiation (see problem 14 for more details).

(2) $(\cos x)\left(\frac{1}{x}\right) + (\ln x)(-\sin x) = (\cos x)\left(\frac{1}{x}\right) + \left(\frac{x}{x}\right)[(\ln x)(-\sin x)]$

$$= \frac{\cos x}{x} + \frac{x(\ln x)(-\sin x)}{x} = \frac{\cos x - x(\ln x)(\sin x)}{x}.$$

(3) Remember to replace y by $x^{\cos x}$ at the end.

(4) See problem 45 for more information on implicit differentiation.

16. Differentiate $f(x) = \frac{e^{\cot 3x}}{\sqrt{x}}$ and express your answer as a simple fraction.

Solution:

$$f'(x) = \frac{\sqrt{x}(e^{\cot 3x})(-\csc^2 3x)(3) - e^{\cot 3x}(\frac{1}{2\sqrt{x}})}{x} = \frac{-6x(\csc^2 3x)e^{\cot 3x} - e^{\cot 3x}}{2x\sqrt{x}}.$$

Notes: (1) $\frac{d}{dx}[e^x] = e^x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[3x] = 3$

$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{x^{\frac{1}{2}}} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

(2) We start off using the quotient rule (see problem 10 for a detailed explanation of the quotient rule). Here we get

$$\frac{\sqrt{x}\cdot\frac{d}{dx}[e^{\cot 3x}] - e^{\cot 3x}\cdot\frac{d}{dx}[\sqrt{x}]}{\left(\sqrt{x}\right)^2}$$

(3) $\frac{d}{dx}[e^{\cot 3x}]$ requires two applications of the chain rule. See problem 12 for a detailed explanation of the chain rule. Here we get

$$\frac{d}{dx}[e^{\cot 3x}] = e^{\cot 3x}(-\csc^2 3x)(3).$$

(4) After differentiating we wind up with a complex fraction:

$$\frac{\sqrt{x}(e^{\cot 3x})(-\csc^2 3x)(3) - e^{\cot 3x}(\frac{1}{2\sqrt{x}})}{x}$$

We simplify this complex fraction by multiplying the numerator and denominator by $2\sqrt{x}$.

Note the following:

$x(2\sqrt{x}) = 2x\sqrt{x}$ (this is where the final denominator comes from).

$\sqrt{x}(e^{\cot 3x})(-\csc^2 3x)(3)(2\sqrt{x}) = -6\sqrt{x}\sqrt{x}\,(\csc^2 3x)e^{\cot 3x} = -6x\,(\csc^2 3x)e^{\cot 3x}$

$e^{\cot 3x}\left(\frac{1}{2\sqrt{x}}\right)(2\sqrt{x}) = e^{\cot 3x}$

The last two results together with the distributive property give the final numerator.

LEVEL 1: INTEGRATION

17. $\int (x^4 - 6x^2 + 3)\,dx =$

(A) $5x^5 - 18x^3 + 3x + C$

(B) $4x^3 - 12x + 3x + C$

(C) $\frac{x^5}{4} - 3x^2 + 3x + C$

(D) $\frac{x^5}{5} - 2x^3 + 3x + C$

Solution:

$$\int (x^4 - 6x^2 + 3)\,dx = \frac{x^5}{5} - \frac{6x^3}{3} + 3x + C = \frac{x^5}{5} - 2x^3 + 3x + C.$$

This is choice (D).

Notes: (1) If n is any real number, then an antiderivative of x^n is $\frac{x^{n+1}}{n+1}$.

Symbolically, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where C is an arbitrary constant.

For example, $\int x^4 dx = \frac{x^5}{5} + C$.

As another example, $\int 3dx = \int 3x^0 dx = \frac{3x^1}{1} + C = 3x + C$.

(2) Of course it is worth just remembering that $\int kdx = kx$ for any constant k.

(3) If g and h are functions, then

$$\int [g(x) + h(x)]dx = \int g(x)dx + \int h(x)dx.$$

In other words, when integrating a sum we can simply integrate term by term.

Similarly, $\int [g(x) - h(x)]dx = \int g(x)dx - \int h(x)dx$.

(4) If g is a function and k is a constant, then

$$\int kg(x)dx = k\int g(x)dx$$

For example, $\int 6x^2 dx = 6\int x^2 dx = 6\left(\frac{x^3}{3}\right) + C = 2x^3 + C$.

(5) In the given problem we integrate each of x^4, x^2, and 3 separately and then use notes (3) and (4) to write the final answer.

(6) We do not need to include a constant C for each individual integration since if we add or subtract two or more constants we simply get a new constant. This is why we simply add one constant C at the end of the integration.

(7) It is also possible to solve this problem by differentiating the answer choices. For example, if we start with choice (C), then we have that $\frac{d}{dx}\left(\frac{x^5}{4}\right) = \frac{5x^4}{4}$. So we can immediately see that choice (C) is incorrect.

When we differentiate choice (D) however, we get

$$\frac{d}{dx}\left[\frac{x^5}{5} - 2x^3 + 3x + C\right] = \frac{5x^4}{5} - 3(2x^2) + 3 + 0 = x^4 - 6x^2 + 3.$$

This is the **integrand** (the expression between the integral symbol and dx) that we started with. So the answer is choice (D).

(8) Note that the derivative of any constant is always 0, ie. $\frac{d}{dx}[C] = 0$.

18. $\int_{-1}^{2}(3x^2 - 2x)\,dx =$

(A) 2
(B) 4
(C) 6
(D) 14

Solution: $\int_{-1}^{2}(3x^2 - 2x)\,dx = (\frac{3x^3}{3} - \frac{2x^2}{2})\Big|_{-1}^{2} = (x^3 - x^2)\,\Big|_{-1}^{2} =$ $(2^3 - 2^2) - ((-1)^3 - (-1)^2) = (8 - 4) - (-1 - 1) = 4 - (-2) = 6$.

This is choice (C).

Notes: (1) $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f.

In this example, $F(x) = x^3 - x^2$ is an antiderivative of the function $f(x) = 3x^2 - 2x$. So $\int_{-1}^{2} f(x)dx = F(2) - F(-1)$

(2) We sometimes write $F(b) - F(a)$ as $F(x)\,\Big|_a^b$.

This is just a convenient way of focusing on finding an antiderivative before worrying about plugging in the **upper** and **lower limits of integration** (these are the numbers b and a, respectively).

(3) For details on how to find an antiderivative here, see the notes at the end of the solution to problem 17.

19. $3\int e^{3x}\,dx =$

(A) $e^{-3x} + C$
(B) $e^{-x} + C$
(C) $e^{x} + C$
(D) $e^{3x} + C$

Solution: $3\int e^{3x}dx = \int e^{3x}(3dx) = e^{3x} + C$, choice (D).

Notes: (1) We can formally make the substitution $u = 3x$. It then follows that $du = 3dx$. So we have

$$\int e^{3x}(3dx) = \int e^{u}du = e^{u} + C = e^{3x} + C.$$

We get the leftmost equality by replacing $3x$ by u, and $3dx$ by du.

We get the second equality by the basic integration formula

$$\int e^u du = e^u + C.$$

And we get the rightmost equality by replacing u with $3x$.

(2) Note that the function $f(x) = e^{3x}$ can be written as the composition $f(x) = g(h(x))$ where $g(x) = e^x$ and $h(x) = 3x$.

Since $h(x) = 3x$ is the inner part of the composition, it is natural to try the substitution $u = 3x$.

Note that the derivative of $3x$ is 3, so that $du = 3dx$.

(3) With a little practice, we can evaluate an integral like this very quickly with the following reasoning: The derivative of $3x$ is 3. So to integrate $3e^{3x}$ we simply pretend we are integrating e^x but as we do it we leave the $3x$ where it is. This is essentially what was done in the above solution.

Note that the 3 “goes away” because it is the derivative of $3x$. We need it there for everything to work.

(4) We can also solve this problem by differentiating the answer choices. In fact, we have $\frac{d}{dx}[e^{3x} + C] = e^{3x}(3) + 0 = 3e^{3x}$. So the answer is choice (D).

20. $\int (x^2 + 2)\sqrt{x}\, dx =$

(A) $x^2 + 4x^{\frac{3}{2}} + x + C$

(B) $\frac{2}{7}x^{\frac{7}{2}} + \frac{4}{3}x^{\frac{3}{2}} + C$

(C) $\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{1}{2}} + C$

(D) $\frac{5}{2}\sqrt{x} + \frac{2}{\sqrt{x}} + c$

Solution:

$$\int (x^2 + 2)\sqrt{x}\, dx = \int (x^2\sqrt{x} + 2\sqrt{x})dx = \int x^2 x^{\frac{1}{2}} dx + \int 2x^{\frac{1}{2}}\, dx$$

$$= \int x^{\frac{5}{2}} dx + \int 2x^{\frac{1}{2}}\, dx = \frac{2}{7}x^{\frac{7}{2}} + \frac{4}{3}x^{\frac{3}{2}} + C$$

This is choice (B).

Notes: (1) $\sqrt{x} = x^{\frac{1}{2}}$. So $2\sqrt{x} = 2x^{\frac{1}{2}}$.

(2) $x^2 x^{\frac{1}{2}} = x^{2+\frac{1}{2}} = x^{\frac{4}{2}+\frac{1}{2}} = x^{\frac{5}{2}}$.

(3) See problem 11 for more information on the laws of exponents used here.

(4) To integrate we used the power rule twice: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

$$\int x^{\frac{1}{2}} dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = x^{\frac{3}{2}} \div \frac{3}{2} + C = x^{\frac{3}{2}} \cdot \frac{2}{3} + C = \frac{2}{3} x^{\frac{3}{2}} + C.$$

It follows that $\int 2x^{\frac{1}{2}} dx = 2\int x^{\frac{1}{2}} dx = 2 \cdot \frac{2}{3} x^{\frac{3}{2}} + C = \frac{4}{3} x^{\frac{3}{2}} + C$.

$$\int x^{\frac{5}{2}} dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + C = x^{\frac{7}{2}} \div \frac{7}{2} + C = x^{\frac{7}{2}} \cdot \frac{2}{7} + C = \frac{2}{7} x^{\frac{7}{2}} + C.$$

21. $\int_0^2 (x^2 - 4x) e^{6x^2 - x^3} dx =$

(A) $-\frac{e^{16}}{3}$

(B) 0

(C) $\frac{e^{16}}{3}$

(D) $\frac{1-e^{16}}{3}$

Solution:

$\int_0^2 (x^2 - 4x) e^{6x^2 - x^3} dx = -\frac{1}{3} e^{6x^2 - x^3} \Big|_0^2 = -\frac{1}{3} e^{16} - \left(-\frac{1}{3} e^0\right) = \frac{-e^{16}+1}{3}$

This is equivalent to choice (D).

Notes: (1) To evaluate $\int (x^2 - 4x) e^{6x^2 - x^3} dx$, we can formally make the substitution $u = 6x^2 - x^3$. It then follows that

$$du = (12x - 3x^2)dx = 3(4x - x^2)dx = -3(x^2 - 4x)dx$$

Uh oh! There is no factor of -3 inside the integral.

But constants never pose a problem. We simply multiply by -3 and $-\frac{1}{3}$ at the same time. We place the -3 inside the integral where it is needed, and we leave the $-\frac{1}{3}$ outside of the integral sign as follows:

$$\int (x^2 - 4x)e^{6x^2 - x^3}dx = -\frac{1}{3}\int(-3)(x^2 - 4x)e^{6x^2-x^3}dx$$

We have this flexibility to place the -3 and $-\frac{1}{3}$ where we like because multiplication is commutative, and constants can be pulled outside of the integral sign freely.

We now have

$$\int (x^2 - 4x)e^{6x^2 - x^3}dx = -\frac{1}{3}\int(-3)(x^2 - 4x)e^{6x^2-x^3}dx$$

$$= -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{6x^2-x^3} + C$$

(2) If we are doing the substitution formally, we can save some time by changing the limits of integration. We do this as follows:

$$\int_0^2 (x^2 - 4x)e^{6x^2 - x^3}\,dx = -\frac{1}{3}\int_0^2(-3)(x^2 - 4x)e^{6x^2-x^3}\,dx$$

$$= -\frac{1}{3}\int_0^{16} e^u\,du = -\frac{1}{3}e^u\,\Big|_0^{16} = -\frac{1}{3}(e^{16} - e^0) = \frac{-e^{16}+1}{3}$$

Notice that the limits 0 and 2 were changed to the limits 0 and 16, respectively. We made this change using the formula that we chose for the substitution: $u = 6x^2 - x^3$. When $x = 0$, we have $u = 0$ and when $x = 2$, we have $u = 6(2)^2 - 2^3 = 6 \cdot 4 - 8 = 24 - 8 = 16$.

22. $\int (\frac{2}{x^2} + \frac{1}{x} - 5\sqrt{x} + \frac{7}{\sqrt[3]{x^5}})\,dx =$

Solution: $\int (\frac{2}{x^2} + \frac{1}{x} - 5\sqrt{x} + \frac{7}{\sqrt[3]{x^5}})\,dx = \int (2x^{-2} + \frac{1}{x} - 5x^{\frac{1}{2}} + 7x^{-\frac{5}{3}}\,dx$

$= \frac{2x^{-1}}{-1} + \ln|x| - \frac{5x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{7x^{-\frac{2}{3}}}{-\frac{2}{3}} + C = -\frac{2}{x} + \ln|x| - \frac{10}{3}\sqrt{x^3} - \frac{21}{2\sqrt[3]{x^2}} + C.$

Notes: (1) Recall from problem 11 that $\frac{d}{dx}[\ln x] = \frac{1}{x}$. It therefore seems like it should follow that $\int \frac{1}{x}dx = \ln x + C$. But this is *not* completely accurate.

Observe that we also have $\frac{d}{dx}[\ln(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}$ (the chain rule was used here). So it appears that we also have $\int \frac{1}{x}\,dx = \ln(-x) + C$.

How can the same integral lead to two different answers? Well it doesn't. Note that $\ln x$ is only defined for $x > 0$, and $\ln(-x)$ is only defined for $x < 0$.

Furthermore, observe that $\ln|x| = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$.

It follows that

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

(2) $\frac{2}{x^2} = 2\left(\frac{1}{x^2}\right) = 2x^{-2}$. It follows that $\int \frac{2}{x^2}\,dx = \int 2x^{-2}\,dx = \frac{2x^{-1}}{-1} + C$.

Also, $\frac{2x^{-1}}{-1} = -2x^{-1} = -2\left(\frac{1}{x}\right) = -\frac{2}{x}$.

(3) $\sqrt{x} = x^{\frac{1}{2}}$. It follows that $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$.

Also, $\frac{x^{\frac{3}{2}}}{\frac{3}{2}} = x^{\frac{3}{2}} \div \frac{3}{2} = x^{\frac{3}{2}} \cdot \frac{2}{3} = \frac{2x^{\frac{3}{2}}}{3} = \frac{2\sqrt{x^3}}{3}$.

(4) $\frac{7}{\sqrt[3]{x^5}} = \frac{7}{x^{\frac{5}{3}}} = 7\left(\frac{1}{x^{\frac{5}{3}}}\right) = 7x^{-\frac{5}{3}}$. It follows that

$$\int \frac{7}{\sqrt[3]{x^5}}\,dx = \int 7x^{-\frac{5}{3}}\,dx = 7\left(\frac{x^{-\frac{2}{3}}}{-\frac{2}{3}}\right) + C.$$

Also, $\frac{x^{-\frac{2}{3}}}{-\frac{2}{3}} = x^{-\frac{2}{3}} \div \left(-\frac{2}{3}\right) = x^{-\frac{2}{3}} \cdot \left(-\frac{3}{2}\right) = \frac{1}{x^{\frac{2}{3}}} \cdot \left(-\frac{3}{2}\right) = -\frac{3}{2x^{\frac{2}{3}}} = -\frac{3}{2\sqrt[3]{x^2}}$.

(5) See problem 11 for more information on the laws of exponents used here.

23. $\int \frac{1}{x \ln x}\,dx =$

Solution: $\int \frac{1}{x \ln x}\,dx = \ln|\ln x| + C$.

Note: To evaluate $\int \frac{1}{x\ln x}\,dx$, we can formally make the substitution $u = \ln x$. It then follows that $du = \frac{1}{x}\,dx$. So we have

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C.$$

To get the first equality we simply rewrote $\frac{1}{x\ln x}$ as $\frac{1}{x}\cdot\frac{1}{\ln x} = \frac{1}{\ln x}\cdot\frac{1}{x}$. This way it is easier to see exactly where u and du are.

To get the second equality we simply replaced $\ln x$ by u, and $\frac{1}{x}\,dx$ by du.

To get the third equality we used the basic integration formula

$$\int \frac{1}{x}\,du = \ln|x| + C \text{ (see problem 22 for details).}$$

To get the last equality we replaced u by $\ln x$ (since we set $u = \ln x$ in the beginning).

24. $\int 5^{\cot x}\csc^2 x\,dx =$

Solution: $\int 5^{\cot x}\csc^2 x\,dx = -\frac{5^{\cot x}}{\ln 5} + C$

Notes: (1) Recall from problem 14 that $\frac{d}{dx}[5^x] = 5^x(\ln 5)$. It follows that $\int 5^x\,dx = \frac{5^x}{\ln 5} + C$.

To verify this, note that

$$\frac{d}{dx}\left[\frac{5^x}{\ln 5} + C\right] = \frac{1}{\ln 5}\frac{d}{dx}[5^x] + \frac{d}{dx}[C] = \frac{1}{\ln 5}\cdot 5^x(\ln 5) + 0 = 5^x.$$

More generally, we have that for any $b > 0$, $b \neq 1$, $\int b^x\,dx = \frac{b^x}{\ln b}$

(2) As an alternative way to evaluate $\int 5^x\,dx$, we can rewrite 5^x as $e^{x\ln 5}$ and perform the substitution $u = x\ln 5$, so that $du = (\ln 5)\,dx$. So we have

$$\int 5^x\,dx = \int e^{x\ln 5}\,dx = \frac{1}{\ln 5}\int e^{x\ln 5}(\ln 5)\,dx = \frac{1}{\ln 5}\int e^u\,du$$

$$= \frac{1}{\ln 5}e^u + C = \frac{1}{\ln 5}e^{x\ln 5} + C = \frac{1}{\ln 5}5^x + C = \frac{5^x}{\ln 5} + C.$$

(3) To evaluate $\int 5^{\cot x}\csc^2 x\,dx$, we can formally make the substitution $u = \cot x$. It then follows that $du = -\csc^2 x\,dx$. So we have

$$\int 5^{\cot x} \csc^2 x\, dx = -\int 5^{\cot x} (-\csc^2 x) dx = -\int 5^u\, du = -\frac{5^u}{\ln 5} + C$$

$$= -\frac{5^{\cot x}}{\ln 5} + C.$$

(4) As an alternative, we can combine notes (2) and (3) to evaluate the integral in a single step by rewriting $5^{\cot x} \csc^2 x$ as $e^{(\cot x)(\ln 5)} \csc^2 x$, and then letting $u = (\cot x)(\ln 5)$, so that $du = (-\csc^2 x)(\ln 5)dx$. I leave the details of this solution to the reader.

LEVEL 1: LIMITS AND CONTINUITY

25. $\lim\limits_{x \to 7} \frac{2x^2 - 13x - 7}{x - 7} =$

(A) ∞
(B) 0
(C) 2
(D) 15

Solution 1: $\lim\limits_{x \to 7} \frac{2x^2-13x-7}{x-7} = \lim\limits_{x \to 7} \frac{(x-7)(2x+1)}{x-7} = \lim\limits_{x \to 7}(2x+1)$

$$= 2(7) + 1 = 15.$$

This is choice (D).

Notes: (1) When we try to substitute 7 in for x we get the **indeterminate form** $\frac{0}{0}$. Here is the computation:

$$\frac{2(7)^2-13(7)-7}{7-7} = \frac{98-91-7}{7-7} = \frac{0}{0}.$$

This means that we cannot use the method of "plugging in the number" to get the answer. So we have to use some other method.

(2) One algebraic "trick" that works in this case is to factor the numerator as $2x^2 - 13x - 7 = (x-7)(2x+1)$. Note that one of the factors is $(x - 7)$ which is identical to the factor in the denominator. This will *always* happen when using this "trick." This makes factoring pretty easy in these problems.

(3) **Most important limit theorem:** If $f(x) = g(x)$ for all x in some interval containing $x = c$ *except* possibly at c itself, then we have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$.

In this problem, our two functions are

$$f(x) = \frac{2x^2 - 13x - 7}{x - 7} \text{ and } g(x) = 2x + 1.$$

Note that f and g agree everywhere *except* at $x = 7$. Also note that $f(7)$ is undefined, whereas $g(7) = 15$.

(4) To compute a limit, first try to simply plug in the number. This will only fail when the result is an indeterminate form. The two **basic** indeterminate forms are $\frac{0}{0}$ and $\frac{\infty}{\infty}$ (the more **advanced** ones are $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0, but these can always be manipulated into one of the two basic forms).

If an indeterminate form results from plugging in the number, then there are two possible options:

Option 1: Use some algebraic manipulations to create a new function that agrees with the original except at the value that is being approached, and then use the limit theorem mentioned in note (3).

This is how we solved the problem above.

Option 2: Try L'Hôpital's rule (see solution 2 below).

Solution 2: We use L'Hôpital's rule to get

$$\lim_{x \to 7} \frac{2x^2 - 13x - 7}{x - 7} = \lim_{x \to 7} \frac{4x - 13}{1} = 4(7) - 13 = 15, \text{ choice (D).}$$

Notes: (1) L'Hôpital's rule says the following: Suppose that

(i) g and k are differentiable on some interval containing c (except possibly at c itself).

(ii) $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} k(x) = 0$ or $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} k(x) = \pm\infty$

(iii) $\lim\limits_{x \to c} \frac{g'(x)}{k'(x)}$ exists, and

(iv) $k'(x) \neq 0$ for all x in the interval (except possibly at c itself).

Then $\lim\limits_{x \to c} \frac{g(x)}{k(x)} = \lim\limits_{x \to c} \frac{g'(x)}{k'(x)}$.

In this problem $g(x) = 2x^2 - 13x - 7$ and $k(x) = x - 7$.

(2) It is very important that we first check that the expression has the correct form before applying L'Hôpital's rule.

In this problem, note that when we substitute 7 in for x in the given expression we get $\frac{0}{0}$ (see note 1 above). So in this case L'Hôpital's rule can be applied.

26. If $h(x) = \frac{5x^2-3x+2}{3x^2-2x}$, then $\lim\limits_{x \to 1} h(x) =$

(A) 4

(B) $\frac{7}{4}$

(C) $\frac{5}{3}$

(D) 0

Solution: $\lim\limits_{x \to 1} h(x) = \lim\limits_{x \to 1} \frac{5x^2-3x+2}{3x^2-2x} = \frac{5(1)^2-3(1)+2}{3(1)^2-2(1)} = \frac{5-3+2}{3-2} = \frac{4}{1} = 4.$

This is choice (A).

Notes: (1) We simply substituted 1 in for x. Since we did *not* get an indeterminate form, we see that the answer is 4.

(2) It would be incorrect to try to apply L'Hôpital's rule here! Let's see what happens if we try:

$$\lim_{x \to 1} \frac{5x^2-3x+2}{3x^2-2x} = \lim_{x \to 1} \frac{10x-3}{6x-2} = \frac{10(1)-3}{6(1)-2} = \frac{7}{4}.$$

So we get choice (D) which is **wrong!**

If we tried to apply L'Hôpital's rule twice we would get

$$\lim_{x \to 1} \frac{5x^2-3x+2}{3x^2-2x} = \lim_{x \to 1} \frac{10x-3}{6x-2} = \lim_{x \to 1} \frac{10}{6} = \frac{5}{3}.$$

So we get choice (C), also **wrong!**

27. $\lim_{x\to\infty} \frac{3x^2-2x+1}{7x^2+5x-3}$

(A) $-\frac{1}{3}$

(B) $\frac{3}{7}$

(C) $\frac{7}{3}$

(D) ∞

Solution: $\lim_{x\to\infty} \frac{3x^2-2x+1}{7x^2+5x-3} = \lim_{x\to\infty} \frac{3x^2}{7x^2} = \frac{3}{7}$, choice (B).

Notes: (1) If p and q are polynomials, then $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$ where

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \text{ and}$$

$$q(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0.$$

(2) If $n = m$, then $\lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m}$.

(3) Combining notes (1) and (2), we could have gotten the answer to this problem immediately by simply taking the coefficients of x^2 in the numerator and denominator and dividing.

The coefficient of x^2 in the numerator is 3, and the coefficient of x^2 in the denominator is 7. So the final answer is $\frac{3}{7}$.

(4) If $n > 0$, then $\lim_{x\to\infty} \frac{1}{x^n} = 0$.

(5) For a more rigorous solution, we can multiply both the numerator and denominator of the fraction by $\frac{1}{x^2}$ to get

$$\frac{3x^2-2x+1}{7x^2+5x-3} = \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} \cdot \frac{(3x^2-2x+1)}{(7x^2+5x-3)} = \frac{3-\frac{2}{x}+\frac{1}{x^2}}{7+\frac{5}{x}-\frac{3}{x^2}}.$$

It follows that $\lim_{x\to\infty} \frac{3x^2-2x+1}{7x^2+5x-3} = \frac{\lim_{x\to\infty} 3 - 2\lim_{x\to\infty}\left(\frac{1}{x}\right) + \lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 7 + 5\lim_{x\to\infty}\left(\frac{1}{x}\right) - 3\lim_{x\to\infty}\left(\frac{1}{x^2}\right)} = \frac{3-2(0)+(0)}{7+5(0)-3(0)} = \frac{3}{7}$.

(6) L'Hôpital's rule can also be used to solve this problem since the limit has the form $\frac{\infty}{\infty}$:

$$\lim_{x\to\infty} \frac{3x^2-2x+1}{7x^2+5x-3} = \lim_{x\to\infty} \frac{6x-2}{14x+5} = \lim_{x\to\infty} \frac{6}{14} = \frac{3}{7}.$$

Observe that we applied L'Hôpital's rule twice. The first time we differentiated the numerator and denominator with respect to x to get another expression of the form $\frac{\infty}{\infty}$.

28. If the function g is continuous for all real numbers and if $g(x) = \frac{x^2-x-6}{x-3}$ for all $x \neq 3$, then $g(3) =$

    (A) 0

    (B) 1

    (C) 2

    (D) 5

Solution 1: $g(3) = \lim\limits_{x\to 3} \frac{x^2-x-6}{x-3} = \lim\limits_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim\limits_{x\to 3}(x+2)$

$$= 3 + 2 = 5.$$

This is choice (D).

Notes: (1) A function f is continuous at $x = a$ if $\lim\limits_{x\to a} f(x) = f(a)$.

In this problem we want the function g to be continuous for all real numbers. In particular, we need g to be continuous at $x = 3$. So we must have $g(3) = \lim\limits_{x\to 3} g(x) = \lim\limits_{x\to 3} \frac{x^2-x-6}{x-3}$.

(2) When we try to substitute 3 in for x we get the **indeterminate form** $\frac{0}{0}$. Here is the computation:

$$\frac{3^2-3-6}{3-3} = \frac{9-9}{3-3} = \frac{0}{0}.$$

This means that we cannot use the method of "plugging in the number" to compute the limit. We used the same method from the first solution in problem 25.

(3) As an alternative we could have used L'Hôpital's rule (just like we did for the second solution in problem 25). Here are the details:

$g(3) = \lim\limits_{x\to 3} \frac{x^2-x-6}{x-3} = \lim\limits_{x\to 3} \frac{2x-1}{1} = 2(3) - 1 = 5$, choice (D).

See problem 25 for more information about L'Hôpital's rule.

29. $\lim_{x \to 0} \frac{\sin^3 5x}{x^3}$

(A) -125

(B) 5

(C) 125

(D) The limit does not exist

Solution: $\lim_{x \to 0} \frac{\sin^3 5x}{x^3} = \lim_{5x \to 0} 5^3 \frac{\sin^3 5x}{5^3 x^3} = 125 \lim_{5x \to 0} \frac{(\sin 5x)^3}{(5x)^3}$

$$= 125 \lim_{u \to 0} \left(\frac{\sin u}{u}\right)^3 = 125(\lim_{u \to 0} \frac{\sin u}{u})^3 = 125(1)^3 = 125.$$

This is choice (C).

Notes: (1) A basic limit worth memorizing is

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

(2) The limit in note (1) is actually very easy to compute using L'Hôpital's rule:

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$

(3) It is not hard to see that $x \to 0$ if and only if $5x \to 0$. This is why we can replace x by $5x$ in the subscript of the limit above.

(4) $\sin^n x$ is an abbreviation for $(\sin x)^n$.

In particular, $\sin^3 x = (\sin x)^3$, and so $\sin^3 5x = (\sin 5x)^3$.

(5) $\frac{\sin 5x}{x}$ can be rewritten as $5 \frac{\sin 5x}{5x}$.

It follows that we can rewrite $\frac{\sin^3 5x}{x^3}$ as $5^3 \frac{\sin^3 5x}{5^3 x^3}$.

(6) $5^3 x^3 = (5x)^3$ by a basic law of exponents, and $\frac{(\sin 5x)^3}{(5x)^3} = \left(\frac{\sin 5x}{5x}\right)^3$ by another basic law of exponents.

(7) Using the substitution $u = 5x$, we have

$$\lim_{5x \to 0} \left(\frac{\sin 5x}{5x}\right)^3 = \lim_{u \to 0} \left(\frac{\sin u}{u}\right)^3.$$

(8) If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, then

$$\lim_{x\to c}[f(x)\,g(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$$

In particular,

$$\lim_{x\to c}[f(x)]^2 = \lim_{x\to c}[f(x)\,f(x)] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} f(x) = [\lim_{x\to c} f(x)]^2$$

More generally, $\lim_{x\to c}[f(x)]^n = [\lim_{x\to c} f(x)]^n$

This is why we have $\lim_{u\to 0}\left(\frac{\sin u}{u}\right)^3 = (\lim_{u\to 0}\frac{\sin u}{u})^3$.

30. $\lim_{x\to 0}\frac{3\tan x - 3\cos^2 x\tan x}{x^3}$

(A) 0

(B) $\frac{1}{3}$

(C) 3

(D) ∞

Solution: $\lim_{x\to 0}\frac{3\tan x - 3\cos^2 x\tan x}{x^3} = \lim_{x\to 0}\frac{3\tan x(1-\cos^2 x)}{x^3}$

$$= \lim_{x\to 0}\frac{3\tan x\sin^2 x}{x^3} = 3\left(\lim_{x\to 0}\frac{\tan x}{x}\right)\cdot(\lim_{x\to 0}\frac{\sin x}{x})^2 = 3(1)(1) = 3.$$

This is choice (C).

Notes: (1) We get the first equality by factoring $3\tan x$.

(2) To get the second equality we used the most basic Pythagorean identity

$$\mathbf{\cos^2 x + \sin^2 x = 1}$$

Subtracting $\cos^2 x$ from each side of this equation gives us

$$\sin^2 x = 1 - \cos^2 x$$

(3) For the third equality we use a basic limit rule concerning products. See problem 29 for details.

(4) As noted in problem 29, $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

We also have $\lim\limits_{x\to 0}\frac{\tan x}{x}=1$.

Both of these results can easily be seen by using L'Hôpital's rule.

For the second one, we have

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\sec^2 x}{1}=\sec^2 0=(\sec 0)^2=\left(\frac{1}{\cos 0}\right)^2=1.$$

31. If the function f is continuous for all x in the interval $[a,b]$, then at any point c in the interval (a,b), which of the following must be true?

(A) $\lim\limits_{x\to c} f(x)=f(c)$

(B) $f'(c)$ exists

(C) $f(c)=0$

(D) $f(c)=f(b)-f(a)$

Solution: f is continuous at $x=c$ if and only if $\lim\limits_{x\to c} f(x)=f(c)$, choice (A).

Notes: (1) This question is simply asking us to pick out the definition of continuity at a value $x=c$.

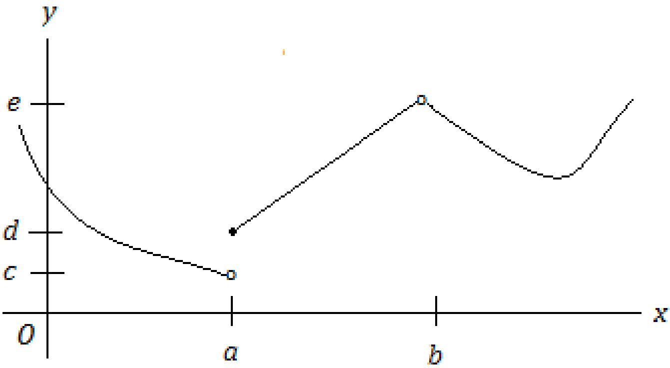
(2) The function $f(x)=|x|$ on the interval $[-1,1]$ is a counterexample to choices (B), (C), and (D).

Note that $f(x)=|x|$ is continuous for all x. In particular, f is continuous for all x in $[-1,1]$.

$f'(0)$ does not exist, since the graph of f has a “sharp edge” there. This allows us to eliminate choice (B).

$$f\left(\frac{1}{2}\right)=\frac{1}{2}\neq 0 \text{ and } f(-1)-f(1)=1-1=0.$$

This allows us to eliminate choices (C) and (D).



32. The graph of the function h is shown in the figure above. Which of the following statements about h is true?

(A) $\lim_{x \to a} h(x) = c$

(B) $\lim_{x \to a} h(x) = d$

(C) $\lim_{x \to b} h(x) = e$

(D) $\lim_{x \to b} h(x) = h(b)$

Solution: From the graph we see that $\lim_{x \to b} h(x) = e$, choice (C).

Notes: (1) The open circles on the graph at a and b indicate that there is no point at that location. The darkened circle at a indicates $h(a) = d$.