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arranged by **Topic** and **Difficulty** Level

By Dr. Steve Warner

320 Level 1, 2, 3, 4, and 5 AP Calculus BC Problems

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Dr. Steve Warner

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Problems by Level and Topic with Fully Explained Solutions

Level 1: Differentiation

1. If $f(x) = 7x^4 + x + 3\pi - \sec x$, then $f'(x) =$

   (A) $28x^3 + 1 - \sec x \tan x$

   (B) $28x^3 + 1 + \sec x \tan x$

   (C) $28x^3 + 3 - \sec x \tan x$

   (D) $\frac{7}{5}x^5 + \frac{x^2}{2} + 3\pi x - \ln|\sec x + \tan x|$

Solution: $f'(x) = 28x^3 + 1 - \sec x \tan x$. This is choice (A).

Notes: (1) If n is any real number, then the derivative of x^n is nx^{n-1}.

Symbolically, $\boldsymbol{\frac{d}{dx}[x^n] = nx^{n-1}}$.

For example, $\frac{d}{dx}[x^4] = 4x^3$.

As another example, $\frac{d}{dx}[x] = \frac{d}{dx}[x^1] = 1x^0 = 1(1) = 1$.

(2) Of course it is worth just remembering that $\frac{d}{dx}[x] = 1$.

(3) The derivative of a constant is 0. A **constant** is just a real number.

For example, 3π is a constant. So $\frac{d}{dx}[3\pi] = 0$.

(4) The derivative of a constant times a function is the constant times the derivative of the function.

Symbolically, $\frac{d}{dx}[cg(x)] = c\frac{d}{dx}[g(x)]$.

For example, $\frac{d}{dx}[7x^4] = 7\frac{d}{dx}[x^4] = 7 \cdot 4x^3 = 28x^3$.

(5) You should know the derivatives of the six basic trig functions:

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

(6) If g and h are functions, then $(g+h)'(x) = g'(x) + h'(x)$.

In other words, when differentiating a sum, we can simply differentiate term by term.

Similarly, $(g-h)'(x) = g'(x) - h'(x)$.

(7) In the given problem we differentiate each of x^4, x, 3π and $\sec x$ separately and then use notes (4) and (6) to write the final answer.

2. If $g(x) = \frac{e^{4x-4}}{4} - \ln(x^2) + (2x-1)^{\frac{5}{2}}$, then $g'(1) =$

(A) 1
(B) 2
(C) 3
(D) 4

Solution: $g'(x) = e^{4x-4} - \frac{2}{x} + 5(2x-1)^{\frac{3}{2}}$.

Therefore $g'(1) = 1 - 2 + 5 = 4$, choice (D).

Notes: (1) The derivative of e^x is e^x.

Symbolically, $\frac{d}{dx}[e^x] = e^x$.

(2) The derivative of $\ln x$ is $\frac{1}{x}$.

Symbolically, $\frac{d}{dx}[\ln x] = \frac{1}{x}$.

(3) In this problem we need the **chain rule** which says the following:

If $f(x) = (g \circ h)(x) = g(h(x))$, then

$$f'(x) = g'\big(h(x)\big) \cdot h'(x)$$

For example, if $f(x) = \ln(x^2)$, then $f(x) = g(h(x))$ where $g(x) = \ln x$ and $h(x) = x^2$. So $f'(x) = g'\big(h(x)\big) \cdot h'(x) = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$.

Similarly, we have $\frac{d}{dx}[\frac{e^{4x-4}}{4}] = \frac{1}{4} \cdot \frac{d}{dx}[e^{4x-4}] = \frac{1}{4}e^{4x-4} \cdot 4 = e^{4x-4}$, and

$\frac{d}{dx}[(2x-1)^{\frac{5}{2}}] = \frac{5}{2}(2x-1)^{\frac{3}{2}}(2) = 5(2x-1)^{\frac{3}{2}}$.

(4) As an alternative to using the chain rule to differentiate $\ln(x^2)$, we can rewrite $\ln(x^2)$ as $2\ln x$. Then $\frac{d}{dx}[2\ln x] = 2\frac{d}{dx}[\ln x] = 2 \cdot \frac{1}{x} = \frac{2}{x}$. See the first table in problem 3 for the rule of logarithms used here.

(5) In the given problem we differentiate each of $\frac{e^{4x-4}}{4}$, $\ln(x^2)$, and $(2x-1)^{\frac{5}{2}}$ separately and then use note (6) from problem 1 to write the final answer.

(6) If we could use a calculator for this problem, we can compute $g'(x)$ at $x = 1$ using our TI-84 calculator by first selecting nDeriv((or pressing 8) under the MATH menu, then typing the following:

e^(4X – 4)/4 – ln(X^2) + (2X – 1)^(5/2), X, 1),

and pressing ENTER. The display will show approximately 4.

3. $\frac{d}{dx}\left[\frac{x \ln e^{x^5}}{6}\right] =$

(A) $6x^5$
(B) x^5
(C) $6x^5 + x^6$
(D) $x^5 + x^6$

Solution: $\ln e^{x^5} = x^5$, so that $\frac{x \ln e^{x^5}}{6} = \frac{x \cdot x^5}{6} = \frac{1}{6}x^6$. Therefore we have $\frac{d}{dx}\left[\frac{x \ln e^{x^5}}{6}\right] = \frac{d}{dx}\left[\frac{1}{6}x^6\right] = \frac{1}{6} \cdot 6x^5 = x^5$, choice (B).

Notes: (1) $f(x) = \log_e x$ is called the *natural logarithmic function* and is usually abbreviated as $f(x) = \ln x$.

(2) Here are two ways to simplify $\ln e^{x^5}$.

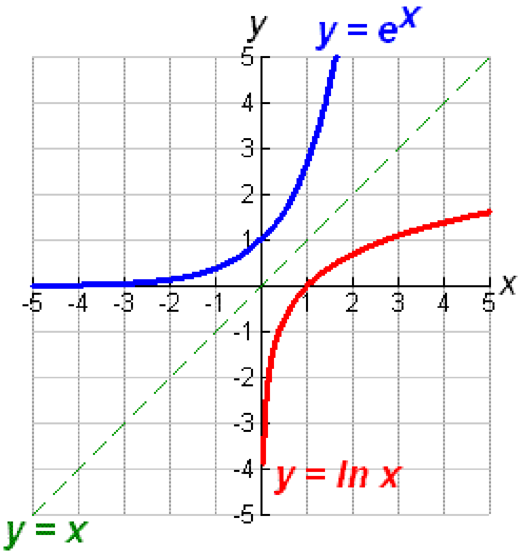
Method 1: Recall that $\ln e = 1$. We have $\ln e^{x^5} = x^5 \ln e = x^5(1) = x^5$. Here we have used the last law in the following table:

Laws of Logarithms: Here is a review of the basic laws of logarithms.

Law	Example
$\log_b 1 = 0$	$\log_2 1 = 0$
$\log_b b = 1$	$\log_6 6 = 1$
$\log_b x + \log_b y = \log_b(xy)$	$\log_5 7 + \log_5 2 = \log_5 14$
$\log_b x - \log_b y = \log_b(\frac{x}{y})$	$\log_3 21 - \log_3 7 = \log_3 3 = 1$
$\log_b x^n = n\log_b x$	$\log_8 3^5 = 5\log_8 3$

Method 2: Recall that the functions e^x and $\ln x$ are inverses of each other. This means that $e^{\ln x} = x$ and $\ln e^x = x$. Replacing x by x^5 in the second equation gives $\ln e^{x^5} = x^5$.

(3) Geometrically inverse functions have graphs that are mirror images across the line $y = x$. Here is a picture of the graphs of $y = e^x$ and $y = \ln x$ together with the line $y = x$. Notice how the line $y = x$ acts as a mirror for the two functions.



(4) $x \cdot x^5 = x^1 \cdot x^5 = x^{1+5} = x^6$.

Here is a complete review of the laws of exponents:

Law	Example
$x^0 = 1$	$3^0 = 1$
$x^1 = x$	$9^1 = 9$
$x^a x^b = x^{a+b}$	$x^3 x^5 = x^8$
$x^a / x^b = x^{a-b}$	$x^{11}/x^4 = x^7$
$(x^a)^b = x^{ab}$	$(x^5)^3 = x^{15}$
$(xy)^a = x^a y^a$	$(xy)^4 = x^4 y^4$
$(x/y)^a = x^a/y^a$	$(x/y)^6 = x^6/y^6$
$x^{-1} = 1/x$	$3^{-1} = 1/3$
$x^{-a} = 1/x^a$	$9^{-2} = 1/81$
$x^{1/n} = \sqrt[n]{x}$	$x^{1/3} = \sqrt[3]{x}$
$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$	$x^{9/2} = \sqrt{x^9} = \left(\sqrt{x}\right)^9$

4. The slope of the tangent line to the graph of $y = xe^{2x}$ at $x = \ln 3$ is

   (A) 9
   (B) 18
   (C) $18 \ln 3$
   (D) $18 \ln 3 + 9$

Solution: $y' = xe^{2x} \cdot 2 + e^{2x} \cdot 1 = 2xe^{2x} + e^{2x} = e^{2x}(2x + 1)$. When $x = \ln 3$, we have that the slope of the tangent line is

$$y'|_{x=\ln 3} = e^{2 \ln 3}(2 \ln 3 + 1) = e^{\ln 3^2}(2 \ln 3 + 1)$$

$$= 3^2(2 \ln 3 + 1) = 9(2 \ln 3 + 1) = 18 \ln 3 + 9.$$

This is choice (D).

Notes: (1) To find the slope of a tangent line to the graph of a function, we simply take the derivative of that function. If we want the slope of the tangent line at a specified x-value, we substitute that x-value into the derivative of the function.

(2) The derivative of $f(x) = e^x$ is $f'(x) = e^x$

(3) In this problem we used the **product rule** which says the following:

If $f(x) = u(x)v(x)$, then

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$

(4) When differentiating e^{2x} we needed to use the chain rule. See problem 2 for details.

(5) See problem 3 for information on logarithms.

(6) The functions e^x and $\ln x$ are inverses of each other. This means that $e^{\ln x} = x$ and $\ln e^x = x$. In particular, $e^{\ln 3^2} = 3^2$.

(7) $n \ln x = \ln x^n$. In particular, $2 \ln 3 = \ln 3^2$. See the first table in problem 3 for the rule of logarithms used here.

(8) Using notes (6) and (7) together we get $e^{2 \ln 3} = e^{\ln 3^2} = e^{\ln 9} = 9$.

(9) As an alternative to using the rule of logarithms as was done in note (8), we can use a law of exponents instead to write $e^{2 \ln 3} = \left(e^{\ln 3}\right)^2$. Since $e^{\ln 3} = 3$, we have $e^{2 \ln 3} = \left(e^{\ln 3}\right)^2 = 3^2 = 9$.

The rule that we used here is $(x^a)^b = x^{ab}$ with $a = \ln 3$ and $b = 2$.

(9) If we could use a calculator for this problem, we can compute y' at $x = \ln 2$ using our TI-84 calculator by first selecting nDeriv((or pressing 8) under the MATH menu, then typing the following: Xe^(2X), X, ln 3), and pressing ENTER. The display will show approximately 28.775.

When we put choice (D) in our calculator we also get approximately 28.775.

5. If $x = \ln(t^2 + 1)$ and $y = \cos 3t$, then $\frac{dy}{dx} =$

(A) $-\frac{3 \sin 3t}{t^2+1}$

(B) $-\frac{3 \sin 3t}{2t(t^2+1)}$

(C) $-\frac{3(t^2+1) \sin 3t}{2t}$

(D) $-\frac{3 \sin 3t}{2t}$

Solution: $\frac{dy}{dt} = -3 \sin 3t$ and $\frac{dx}{dt} = \frac{2t}{t^2+1}$. Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = (-3 \sin 3t) \div \frac{2t}{t^2+1} = (-3 \sin 3t) \cdot \frac{t^2+1}{2t} = -\frac{3(t^2+1) \sin 3t}{2t}.$$

This is choice (C).

Notes: (1) In this problem we are given a parametrically defined curve. The variable t is called the **parameter**, and the two given equations are called **parametric equations**.

For example, when $t = 0$, we have that $x = \ln(0^2 + 1) = \ln 1 = 0$ and $y = \cos(3 \cdot 0) = 1$. So the point $(0,1)$ is on the given parametrically defined curve, and this point corresponds to the parameter value $t = 0$.

Each value for t corresponds to a point (x, y) in the xy-plane.

(2) The derivative $\frac{dy}{dx}$ is equal to $\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

(3) The derivative of $\ln x$ is $\frac{1}{x}$.

Symbolically, $\frac{d}{dx}[\ln x] = \frac{1}{x}$.

(4) The derivatives $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both required the chain rule. See problem 2 for a detailed explanation of this rule.

6. If $g(x) = \pi e^3 + \frac{1}{\sqrt[3]{x^2}} + \left(\frac{x+2}{x-2}\right)^2 - 11^x$, then $g'(x) =$

Solution: We first rewrite g as $g(x) = \pi e^3 + x^{-\frac{2}{3}} + \left(\frac{x+2}{x-2}\right)^2 - 11^x$.

$$g'(x) = 0 - \frac{2}{3}x^{-\frac{5}{3}} + 2\left(\frac{x+2}{x-2}\right)\frac{(x-2)(1) - (x+2)(1)}{(x-2)^2} - 11^x(\ln 11)$$

$$= -\frac{2}{3\sqrt[3]{x^5}} - 8\frac{x+2}{(x-2)^3} - (\ln 11)11^x.$$

Notes: (1) πe^3 is a constant. Therefore $\frac{d}{dx}[\pi e^3] = 0$.

(2) $\frac{1}{\sqrt[3]{x^2}} = \frac{1}{x^{\frac{2}{3}}} = x^{-\frac{2}{3}}$. So $\frac{d}{dx}\left[\frac{1}{\sqrt[3]{x^2}}\right] = \frac{d}{dx}\left[x^{-\frac{2}{3}}\right] = -\frac{2}{3}x^{-\frac{2}{3}-1} = -\frac{2}{3}x^{-\frac{5}{3}}$.

(3) $-\frac{2}{3}x^{-\frac{5}{3}} = -\frac{2}{3x^{\frac{5}{3}}} = -\frac{2}{3\sqrt[3]{x^5}}$.

(4) The **quotient rule** says the following:

If $f(x) = \frac{N(x)}{D(x)}$, then

$$f'(x) = \frac{D(x)N'(x) - N(x)D'(x)}{[D(x)]^2}$$

I like to use the letters N for "numerator" and D for "denominator."

(5) The derivative of $x + 2$ is 1 because the derivative of x is 1, and the derivative of any constant is 0.

Similarly, the derivative of $x - 2$ is also 1.

Now using the quotient rule we see that the derivative of $\frac{x+2}{x-2}$ is $\frac{(x-2)(1)-(x+2)(1)}{(x-2)^2} = \frac{x-2-x-2}{(x-2)^2} = \frac{-4}{(x-2)^2}$.

(6) Differentiating $\left(\frac{x+2}{x-2}\right)^2$ requires the chain rule. Using note (5) we see that this derivative is $2\left(\frac{x+2}{x-2}\right)\left(\frac{-4}{(x-2)^2}\right) = -8\frac{x+2}{(x-2)^3}$.

(7) If $b > 0$, then $\frac{d}{dx}[b^x] = b^x(\ln b)$.

In particular, $\frac{d}{dx}[11^x] = 11^x(\ln 11)$.

(8) For $b > 0$, $b^x = e^{x \ln b}$.

To see this, first observe that $e^{x \ln b} = e^{\ln b^x}$ by the power rule for logarithms (see problem 3 for the laws of logarithms).

Second, recall that the functions e^x and $\ln x$ are inverses of each other. This means that $e^{\ln x} = x$ and $\ln e^x = x$. Replacing x by b^x in the first formula yields $e^{\ln b^x} = b^x$.

(9) The formula in note (8) gives an alternate method for differentiating 11^x. We can rewrite 11^x as $e^{x \ln 11}$ and use the chain rule. Here are the details:

$$\frac{d}{dx}[11^x] = \frac{d}{dx}\left[e^{x \ln 11}\right] = e^{x \ln 11}(\ln 11) = 11^x(\ln 11).$$

Note that in the last step we rewrote $e^{x \ln 11}$ as 11^x.

(10) There is one more method we can use to differentiate 11^x. We can use **logarithmic differentiation**.

We start by writing $y = 11^x$.

We then take the natural log of each side of this equation: $\ln y = \ln 11^x$.

We now use the power rule for logarithms to bring the x out of the exponent: $\ln y = x \ln 11$.

Now we differentiate implicitly to get $\frac{1}{y} \cdot \frac{dy}{dx} = \ln 11$.

Solve for $\frac{dy}{dx}$ by multiplying each side of the last equation by y to get $\frac{dy}{dx} = y \ln 11$.

Finally, replacing y by 11^x gives us $\frac{dy}{dx} = 11^x(\ln 11)$.

(11) **Logarithmic differentiation** is a general method that can often be used to handle expressions that have exponents with variables.

(12) See problem 35 for more information on implicit differentiation.

7. Differentiate $f(x) = \frac{e^{\cot 3x}}{\sqrt{x}}$ and express your answer as a simple fraction.

Solution:

$$f'(x) = \frac{\sqrt{x}(e^{\cot 3x})(-\csc^2 3x)(3) - e^{\cot 3x}(\frac{1}{2\sqrt{x}})}{x} = \frac{-6x(\csc^2 3x)e^{\cot 3x} - e^{\cot 3x}}{2x\sqrt{x}}.$$

Notes: (1) $\frac{d}{dx}[e^x] = e^x$

$\frac{d}{dx}[\cot x] = -\csc^2 x$

$\frac{d}{dx}[3x] = 3$

$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{\frac{1}{2}}\right] = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

(2) We start off using the quotient rule (see problem 6 for a detailed explanation of the quotient rule). Here we get

$$\frac{\sqrt{x} \cdot \frac{d}{dx}[e^{\cot 3x}] - e^{\cot 3x} \cdot \frac{d}{dx}[\sqrt{x}]}{\left(\sqrt{x}\right)^2}$$

(3) $\frac{d}{dx}[e^{\cot 3x}]$ requires two applications of the chain rule. See problem 2 for a detailed explanation of the chain rule. Here we get

$$\frac{d}{dx}[e^{\cot 3x}] = e^{\cot 3x}(-\csc^2 3x)(3).$$

(4) After differentiating we wind up with a complex fraction:

$$\frac{\sqrt{x}(e^{\cot 3x})(-\csc^2 3x)(3) - e^{\cot 3x}\left(\frac{1}{2\sqrt{x}}\right)}{x}$$

We simplify this complex fraction by multiplying the numerator and denominator by $2\sqrt{x}$.

Note the following:

$x(2\sqrt{x}) = 2x\sqrt{x}$ (this is where the final denominator comes from).

$\sqrt{x}(e^{\cot 3x})(-\csc^2 3x)(3)(2\sqrt{x}) = -6\sqrt{x}\sqrt{x}\,(\csc^2 3x)e^{\cot 3x} = -6x\,(\csc^2 3x)e^{\cot 3x}$

$e^{\cot 3x}\left(\frac{1}{2\sqrt{x}}\right)(2\sqrt{x}) = e^{\cot 3x}$

The last two results give the final numerator.

8. If $\boldsymbol{F}$ is the vector-valued function defined by $\boldsymbol{F}(t) = \langle\frac{\ln t}{t}, \cos^2 t\rangle$, then $\boldsymbol{F}''(t) =$

Solution:

$\boldsymbol{F}'(t) = \langle\frac{t\left(\frac{1}{t}\right)-(\ln t)(1)}{t^2}, 2\,(\cos t)(-\sin t)\rangle = \langle\frac{1-\ln t}{t^2}, -2\cos t\sin t\rangle$, and so $\boldsymbol{F}''(t) = \langle\frac{t^2\left(\frac{-1}{t}\right)-(1-\ln t)(2t)}{t^4}, -2\cos t\cos t - 2(\sin t)(-\sin t)\rangle$

$= \langle\frac{-t-2t+2t\ln t}{t^4}, -2\,(\cos^2 t - \sin^2 t)\rangle = \boldsymbol{\langle\frac{2\ln t-3}{t^3}, -2\cos 2t\rangle}$.

Notes: (1) A 2-dimensional **vector-valued function** $\boldsymbol{F}$ has the form $\boldsymbol{F}(t) = \langle x(t), y(t)\rangle$ where x and y are ordinary functions of the variable t.

A vector-valued function is just a convenient way to give a parametrically defined curve with a single function.

The vector-valued function given in the problem is equivalent to the parametric equations

$$x = \frac{\ln t}{t}, y = \cos^2 t$$

Can you express the parametric equations given in problem 5 as a vector-valued function?

(2) The derivative of the vector-valued function $\boldsymbol{F}$ which is defined by $\boldsymbol{F}(t) = \langle x(t), y(t)\rangle$ is the vector-valued function $\boldsymbol{F}'$ which is defined by $\boldsymbol{F}'(t) = \langle x'(t), y'(t)\rangle$. In other words, we simply differentiate each component.

In this problem we have $x(t) = \frac{\ln t}{t}$ and $y(t) = \cos^2 t$.

Note also that $\boldsymbol{F}''(t) = \langle x''(t), y''(t)\rangle$.

(3) Recall from problem 5 that $\frac{d}{dx}[\ln x] = \frac{1}{x}$.

(4) We used the quotient rule to differentiate x and x'. See problem 6 for a detailed explanation of the quotient rule.

(5) $\cos^2 t$ is an abbreviation for $(\cos t)^2$. To differentiate y therefore required the chain rule.

(6)To differentiate y' we used the product rule.

(7) The following two identities can be useful:

$$\sin 2t = 2\sin t\cos t \qquad\qquad \cos 2t = \cos^2 t - \sin^2 t$$

The second identity was used when simplifying $y''(t)$.

We could have used the first identity to write

$$y'(t) = -2\cos t\sin t = -2\sin t\cos t = -\sin 2t.$$

Differentiating this last expression then gives

$$y''(t) = -2\cos 2t.$$

LEVEL 1: INTEGRATION

9. $\int(3x^2 - 6\sqrt{x} + e^x)\,dx =$

(A) $6x - \frac{3}{\sqrt{x}} + e^x + C$

(B) $x^3 - 4\sqrt{x^3} + e^x + C$

(C) $x^3 - 3x + e^x + C$

(D) $x^3 - 3x + xe^{x-1} + C$

Solution:

$\int(3x^2 - 6\sqrt{x} + e^x)\,dx = 3 \cdot \frac{x^3}{3} - \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + e^x + C = x^3 - 4\sqrt{x^3} + e^x + C$

This is choice (B).

Notes: (1) If n is any real number, then an antiderivative of x^n is $\frac{x^{n+1}}{n+1}$.

Symbolically, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where C is an arbitrary constant.

For example, $\int x^2 dx = \frac{x^3}{3} + C$.

As another example,

$$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = x^{\frac{3}{2}} \div \frac{3}{2} + C = x^{\frac{3}{2}} \cdot \frac{2}{3} + C = \frac{2}{3}x^{\frac{3}{2}} + C.$$

(2) Since $\frac{d}{dx}[e^x] = e^x$, it follows that $\int e^x dx = e^x + C$

(3) If g and h are functions, then

$$\int[g(x) + h(x)]dx = \int g(x)dx + \int h(x)dx.$$

In other words, when integrating a sum we can simply integrate term by term.

Similarly, $\int[g(x) - h(x)]dx = \int g(x)dx - \int h(x)dx$.

(4) If g is a function and k is a constant, then

$$\int kg(x)dx = k\int g(x)dx$$

For example, $\int 3x^2 dx = 3\int x^2 dx = 3\left(\frac{x^3}{3}\right) + C = x^3 + C$.

(5) In the given problem we integrate each of x^2, $\sqrt{x}$, and e^x separately and then use notes (3) and (4) to write the final answer.

(6) We do not need to include a constant C for each individual integration since if we add or subtract two or more constants we simply get a new constant. This is why we simply add one constant C at the end of the integration.

(7) It is also possible to solve this problem by differentiating the answer choices. For example, if we start with choice (C), then we have that $\frac{d}{dx}(x^3 - 3x + e^x + C) = 3x^2 - 3 + e^x$. So we can immediately see that choice (C) is incorrect.

When we differentiate choice (B) however, we get

$$\frac{d}{dx}\left[x^3 - 4\sqrt{x^3} + e^x + C\right] = \frac{d}{dx}\left[x^3 - 4x^{\frac{3}{2}} + e^x + C\right]$$

$$= 3x^2 - 4\left(\frac{3}{2}x^{\frac{1}{2}}\right) + e^x + 0 = 3x^2 - 6\sqrt{x} + e^x.$$

This is the **integrand** (the expression between the integral symbol and dx) that we started with. So the answer is choice (B).

(8) Note that the derivative of any constant is always 0, ie. $\frac{d}{dx}[C] = 0$.

10. $\int_0^2 (x^2 - 4x)e^{6x^2 - x^3} dx =$

(A) $-\frac{e^{16}}{3}$

(B) 0

(C) $\frac{e^{16}}{3}$

(D) $\frac{1-e^{16}}{3}$

Solution:

$\int_0^2 (x^2 - 4x)e^{6x^2 - x^3} dx = -\frac{1}{3}e^{6x^2 - x^3}\,\Big|_0^2 = -\frac{1}{3}e^{16} - \left(-\frac{1}{3}e^0\right) = \frac{-e^{16}+1}{3}$

This is equivalent to choice (D).

Notes: (1) To evaluate $\int (x^2 - 4x)e^{6x^2 - x^3} dx$, we can formally make the substitution $u = 6x^2 - x^3$. It then follows that

$$du = (12x - 3x^2)dx = 3(4x - x^2)dx = -3(x^2 - 4x)dx$$

Uh oh! There is no factor of -3 inside the integral. But constants never pose a problem. We simply multiply by -3 and $-\frac{1}{3}$ at the same time. We place the -3 inside the integral where it is needed, and we leave the $-\frac{1}{3}$ outside of the integral sign as follows:

$$\int (x^2-4x)e^{6x^2-x^3}dx = -\frac{1}{3}\int(-3)(x^2-4x)e^{6x^2-x^3}dx$$

We have this flexibility to place the -3 and $-\frac{1}{3}$ where we like because multiplication is commutative, and constants can be pulled outside of the integral sign freely.

We now have

$$\int (x^2-4x)e^{6x^2-x^3}dx = -\frac{1}{3}\int(-3)(x^2-4x)e^{6x^2-x^3}dx$$

$$= -\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C = -\frac{1}{3}e^{6x^2-x^3} + C$$

(2) $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f.

In this example, $F(x) = -\frac{1}{3}e^{6x^2-x^3}$ is an antiderivative of the function $f(x) = (x^2-4x)e^{6x^2-x^3}$. So

$$\int_0^2 f(x)dx = F(2) - F(0) = -\frac{1}{3}e^{6\cdot 2^2-2^3} - (-\frac{1}{3}e^0).$$

(3) We sometimes write $F(b) - F(a)$ as $F(x)\,\Big|_a^b$.

This is just a convenient way of focusing on finding an antiderivative before worrying about plugging in the **upper** and **lower limits of integration** (these are the numbers b and a, respectively).

(4) If we are doing the substitution formally, we can save some time by changing the limits of integration. We do this as follows:

$$\int_0^2 (x^2-4x)e^{6x^2-x^3}\,dx = -\frac{1}{3}\int_0^2(-3)(x^2-4x)e^{6x^2-x^3}\,dx$$

$$= -\frac{1}{3}\int_0^{16} e^u\,du = -\frac{1}{3}e^u\,\Big|_0^{16} = -\frac{1}{3}(e^{16}-e^0) = \frac{-e^{16}+1}{3}$$

Notice that the limits 0 and 2 were changed to the limits 0 and 16, respectively. We made this change using the formula that we chose for the substitution: $u = 6x^2 - x^3$. When $x = 0$, we have $u = 0$ and when $x = 2$, we have $u = 6(2)^2 - 2^3 = 6\cdot 4 - 8 = 24 - 8 = 16$.

11. $\int \frac{1}{x\ln x}dx =$

Solution: $\int \frac{1}{x\ln x}dx = \ln|\ln x| + C.$

Notes: (1) To evaluate $\int \frac{1}{x\ln x}dx$, we can formally make the substitution $u = \ln x$. It then follows that $du = \frac{1}{x}dx$. So we have

$$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln x}\cdot\frac{1}{x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C.$$

To get the first equality we simply rewrote $\frac{1}{x\ln x}$ as $\frac{1}{x}\cdot\frac{1}{\ln x} = \frac{1}{\ln x}\cdot\frac{1}{x}$. This way it is easier to see exactly where u and du are.

To get the second equality we simply replaced $\ln x$ by u, and $\frac{1}{x}dx$ by du.

To get the third equality we used the basic integration formula

$$\int \frac{1}{x}dx = \ln|x| + C.$$

To get the last equality we replaced u by $\ln x$ (since we set $u = \ln x$ in the beginning).

(2) Recall from problem 5 that $\frac{d}{dx}[\ln x] = \frac{1}{x}$. It therefore seems like it should follow that $\int \frac{1}{x}dx = \ln x + C$. But this is *not* completely accurate.

Observe that we also have $\frac{d}{dx}[\ln(-x)] = \frac{1}{-x}(-1) = \frac{1}{x}$ (the chain rule was used here). So it appears that we also have $\int \frac{1}{x}dx = \ln(-x) + C$.

How can the same integral lead to two different answers? Well it doesn't. Note that $\ln x$ is only defined for $x > 0$, and $\ln(-x)$ is only defined for $x < 0$.

Furthermore, observe that $\ln|x| = \begin{cases}\ln x & \text{if } x > 0\\ \ln(-x) & \text{if } x < 0\end{cases}$.

It follows that

$$\int \frac{1}{x}dx = \ln|x| + C.$$

12. $\int 5^{\cot x}\csc^2 x\,dx =$

Solution: $\int 5^{\cot x}\csc^2 x\,dx = -\frac{5^{\cot x}}{\ln 5} + C.$

Notes: (1) Recall from problem 6 that $\frac{d}{dx}[5^x] = 5^x(\ln 5)$. It follows that $\int 5^x\,dx = \frac{5^x}{\ln 5} + C$.

To verify this, note that

$$\frac{d}{dx}\left[\frac{5^x}{\ln 5}+C\right]=\frac{1}{\ln 5}\frac{d}{dx}[5^x]+\frac{d}{dx}[C]=\frac{1}{\ln 5}\cdot 5^x(\ln 5)+0=5^x.$$

More generally, we have that for any $b>0$, $b\neq 1$, $\int b^x\,dx=\frac{b^x}{\ln b}+C$.

(2) As an alternative way to evaluate $\int 5^x\,dx$, we can rewrite 5^x as $e^{x\ln 5}$ and perform the substitution $u=x\ln 5$, so that $du=(\ln 5)\,dx$. So we have

$$\int 5^x\,dx=\int e^{x\ln 5}\,dx=\frac{1}{\ln 5}\int e^{x\ln 5}(\ln 5)\,dx=\frac{1}{\ln 5}\int e^u\,du$$

$$=\frac{1}{\ln 5}e^u+C=\frac{1}{\ln 5}e^{x\ln 5}+C=\frac{1}{\ln 5}5^x+C=\frac{5^x}{\ln 5}+C.$$

(3) To evaluate $\int 5^{\cot x}\csc^2 x\,dx$, we can formally make the substitution $u=\cot x$. It then follows that $du=-\csc^2 x\,dx$. So we have

$$\int 5^{\cot x}\csc^2 x\,dx=-\int 5^{\cot x}(-\csc^2 x)dx=-\int 5^u\,du=-\frac{5^u}{\ln 5}+C$$

$$=-\frac{5^{\cot x}}{\ln 5}+C.$$

(4) As an alternative, we can combine notes (2) and (3) to evaluate the integral in a single step by rewriting $5^{\cot x}\csc^2 x$ as $e^{(\cot x)(\ln 5)}\csc^2 x$, and then letting $u=(\cot x)(\ln 5)$, so that $du=(-\csc^2 x)(\ln 5)dx$. I leave the details of this solution to the reader.

13. If f is a continuous function for all real x, and g is an antiderivative of f, then $\lim_{h\to 0}\frac{1}{h}\int_c^{c+h}f(x)\,dx$ is

    (A) $g(0)$
    (B) $g'(0)$
    (C) $g(c)$
    (D) $g'(c)$

Solution: $\lim_{h\to 0}\frac{1}{h}\int_c^{c+h}f(x)\,dx=\lim_{h\to 0}\frac{1}{h}[g(x)]_c^{c+h}$

$$=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}=g'(c).$$

This is choice (D).

Notes: (1) The second Fundamental Theorem of Calculus says that if f is a Riemann integrable function on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f.

In this problem, since g is an antiderivative of f, we have $\int_c^{c+h} f(x)\,dx = g(c+h) - g(c)$.

(2) We sometimes use the notation $[F(x)]_a^b$ as an abbreviation for $F(b) - F(a)$.

This is just a convenient way of focusing on finding an antiderivative before worrying about plugging in the **upper** and **lower limits of integration** (these are the numbers b and a, respectively).

In the problem above we have

$$\int_c^{c+h} f(x)\,dx = [g(x)]_c^{c+h} = g(c+h) - g(c)$$

(3) If a function f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.

(4) Recall the definition of the derivative:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

So we have $g'(c) = \lim_{h\to 0} \frac{g(c+h)-g(c)}{h}$

14. If the function g given by $g(x) = \sqrt{x^3}$ has an average value of 2 on the interval $[0, b]$, then $b =$

(A) $5^{\frac{3}{2}}$
(B) 5
(C) $5^{\frac{2}{3}}$
(D) $5^{\frac{1}{2}}$

Solution: The average value of g on $[0, b]$ is

$$\frac{1}{b-0}\int_0^b x^{\frac{3}{2}}\,dx = \frac{2}{5b}x^{\frac{5}{2}}\Big|_0^b = \frac{2}{5b}\cdot b^{\frac{5}{2}} = \frac{2}{5}b^{\frac{3}{2}}.$$

So we have $\frac{2}{5}b^{\frac{3}{2}} = 2$. Therefore $b^{\frac{3}{2}} = 5$, and so $b = 5^{\frac{2}{3}}$, choice (C).

Notes: (1) The **average value** of the function f over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

(2) Recall from problem 9 that for any real number n, we have $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where C is an arbitrary constant.

For example, $\int x^{\frac{3}{2}}dx = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C = x^{\frac{5}{2}} \div \frac{5}{2} + C = x^{\frac{5}{2}} \cdot \frac{2}{5} + C = \frac{2}{5}x^{\frac{5}{2}} + C$.

(3) $\int_a^b f(x)dx = F(b) - F(a)$ where F is any antiderivative of f.

Here, $G(x) = \frac{2}{5}x^{\frac{5}{2}}$ is an antiderivative of the function $g(x) = x^{\frac{3}{2}}$. So $\int_0^b g(x)dx = G(b) - G(0) = \frac{2}{5}b^{\frac{5}{2}} - 0 = \frac{2}{5}b^{\frac{5}{2}}$.

(4) $\frac{1}{b} \cdot b^{\frac{5}{2}} = b^{-1} \cdot b^{\frac{5}{2}} = b^{-1+\frac{5}{2}} = b^{-\frac{2}{2}+\frac{5}{2}} = b^{\frac{3}{2}}$.

It follows that $\frac{2}{5b} \cdot b^{\frac{5}{2}} = \frac{2}{5} \cdot \frac{1}{b} \cdot b^{\frac{5}{2}} = \frac{2}{5}b^{\frac{3}{2}}$.

(5) We solve the equation $\frac{2}{5}b^{\frac{3}{2}} = 2$ by first multiplying each side of the equation by $\frac{5}{2}$. Since $\frac{5}{2} \cdot \frac{2}{5} = 1$, we get $b^{\frac{3}{2}} = 2\left(\frac{5}{2}\right) = 5$.

We then raise each side of this last equation to the power $\frac{2}{3}$. Since $(b^{\frac{3}{2}})^{\frac{2}{3}} = b^{\frac{3}{2}\cdot\frac{2}{3}} = b^1 = b$, we get $b = 5^{\frac{2}{3}}$.

(6) See problem 3 for a review of the laws of exponents used in notes (4) and (5).

15. $\int_0^\infty 2xe^{-x^2}\,dx$ is

(A) divergent

(B) -1

(C) $\frac{1}{2}$

(D) 1

Solution: $\int_0^\infty 2xe^{-x^2}\,dx = -e^{-x^2}\,|_0^\infty = 0-(-1) = 1$, choice (D).

Notes: (1) The given integral is an **improper integral** because one of the limits of integration is ∞. This is actually a **Type II improper integral**. For an example of a Type I improper integral, see problem 45.

(2) $\int_0^\infty f(x)\,dx$ is an abbreviation for $\lim\limits_{b\to\infty}\int_0^b f(x)\,dx$, and $F(x)\,|_0^\infty$ is an abbreviation for $\lim\limits_{b\to\infty} F(x)\,|_0^b$.

In this problem, $f(x) = 2xe^{-x^2}$ and $F(x) = -e^{-x^2}$.

(3) To evaluate the integral $\int 2xe^{-x^2}\,dx$, we can formally make the substitution $u = -x^2$. It then follows that $du = -2xdx$.

Uh oh! There is no minus sign inside the integral. But constants never pose a problem. We simply multiply by -1 inside the integral where it is needed, and also outside of the integral sign as follows:

$$\int 2xe^{-x^2}dx = -\int -2xe^{-x^2}dx$$

We have this flexibility to do this because constants can be pulled outside of the integral sign freely, and $(-1)(-1) = 1$, so that the two integrals are equal in value.

We now have

$$-\int -2xe^{-x^2}dx = -\int e^u du = -e^u + C = -e^{-x^2} + C.$$

We get the leftmost equality by replacing $-x^2$ by u, and $-2xdx$ by du.

We get the second equality by the basic integration formula

$$\int e^u du = e^u + C.$$

And we get the rightmost equality by replacing u with $-x^2$.

(4) Note that the function $f(x) = e^{-x^2}$ can be written as the composition $f(x) = g(h(x))$ where $g(x) = e^x$ and $h(x) = -x^2$.

Since $h(x) = -x^2$ is the inner part of the composition, it is natural to try the substitution $u = -x^2$.

Note that the derivative of $-x^2$ is $-2x$, so that $du = -2xdx$.

(5) With a little practice, we can evaluate an integral like this very quickly with the following reasoning: The derivative of $-x^2$ is $-2x$. So to integrate $-2xe^{-x^2}$ we simply pretend we are integrating e^x but as we do it we leave the $-x^2$ where it is. This is essentially what was done in the above solution.

Note that the $-2x$ "goes away" because it is the derivative of $-x^2$. We need it there for everything to work.

(6) If we are doing the substitution formally, we can save some time by changing the limits of integration. We do this as follows:

$$\int_0^\infty 2xe^{-x^2}\,dx = -\int_0^\infty -2xe^{-x^2}\,dx$$

$$= -\int_0^{-\infty} e^u\,du = -e^u\,\Big|_0^{-\infty} = -(0 - e^0) = -(-1) = 1.$$

Notice that the limits 0 and ∞ were changed to the limits 0 and $-\infty$, respectively. We made this change using the formula that we chose for the substitution: $u = -x^2$. When $x = 0$, we have that $u = 0$ and when $x = \infty$, we have "$u = -\infty^2 = -\infty \cdot \infty = -\infty$."

I used quotation marks in that last computation because the computation $\infty \cdot \infty$ is not really well-defined. What we really mean is that if we have two expressions that are approaching ∞, then their product is approaching ∞ as well. For all practical purposes, the following computations are valid:

$$\infty \cdot \infty = \infty \qquad \infty + \infty = \infty \qquad -\infty - \infty = -\infty$$

For example, if $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty}[f(x) \cdot g(x)] = \infty$ and $\lim_{x\to\infty}[f(x) + g(x)] = \infty$.

Note that the following forms are **indeterminate**:

$$\frac{\infty}{\infty} \qquad \frac{0}{0} \qquad 0 \cdot \infty \qquad \infty - \infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0$$

For example, if $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then in general we cannot say anything about $\lim_{x\to\infty} \frac{f(x)}{g(x)}$. The value of this limit depends on the specific functions f and g.

16. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = \arctan(xy)$ with the initial condition $f(0) = 2$. What is the approximation of $f(1)$ if Euler's method is used, starting at $x = 0$ with a step size of 0.5?

(A) 1
(B) 2
(C) $2 + \frac{\pi}{8}$
(D) $2 + \frac{\pi}{4}$

Solution: Let's make a table:

(x, y)	dx	$\frac{dy}{dx}$	$dx\left(\frac{dy}{dx}\right) = dy$	$(x + dx, y + dy)$
$(0,2)$	$.5$	0	0	$(.5,2)$
$(.5,2)$	$.5$	$\frac{\pi}{4}$	$\frac{\pi}{8}$	$(1,2 + \frac{\pi}{8})$

From the last entry of the table we see that $f(1) \approx 2 + \frac{\pi}{8}$, choice (C).

Notes: (1) **Euler's method** is a procedure for approximating the solution of a differential equation.

(2) To use Euler's method we must be given a differential equation $\frac{dy}{dx} = f(x, y)$, an initial condition $f(x_0) = y_0$, and a step size dx.

In this problem, we have $\frac{dy}{dx} = \arctan(xy)$, $f(0) = 2$, and $dx = 0.5$.

(3) The initial condition $f(x_0) = y_0$ is equivalent to saying that the point (x_0, y_0) is on the solution curve.

So in this problem we are given that $(0,2)$ is on the solution curve.

(4) We can get an approximation to $f(x_0 + dx)$ by using a table (as shown in the above solution) as follows:

In the first column we put the point (x_0, y_0) as given by the initial condition.

In the second column we put the step size dx.

In the third column we plug the point (x_0, y_0) into the differential equation to get $\frac{dy}{dx}$.

In the fourth column we multiply the numbers in the previous two columns to get dy.

In the fifth column we add dx to x_0 and dy to y_0 to get the point $(x_0 + dx, y_0 + dy)$. This is equivalent to $f(x_0 + dx) = y_0 + dy$.

(5) We can now copy the point from the fifth column into the first column of the next row, and repeat this procedure to approximate $f(x_0 + 2dx)$.

In this problem, since $x_0 = 0$ and $dx = 0.5$, we have $x_0 + 2dx = 1$, and so we are finished after the second iteration of the procedure.

17. The area of the region bounded by the lines $x = 1$, $x = 4$, and $y = 0$ and the curve $y = e^{3x}$ is

(A) $\frac{1}{3}e^3(e^9 - 1)$
(B) $e^3(e^9 - 1)$
(C) $e^{12} - 1$
(D) $3e^3(e^9 - 1)$

Solution: $\int_1^4 e^{3x}\,dx = \frac{1}{3}e^{3x}\,\Big|_1^4 = \frac{1}{3}e^{12} - \frac{1}{3}e^3 = \frac{1}{3}e^3(e^9 - 1)$.

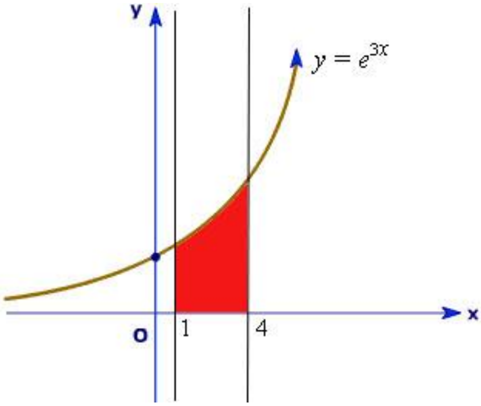
This is choice (A).

Notes: (1) To compute the area under the graph of a function that lies entirely above the x-axis (the line $y = 0$) from $x = a$ to $x = b$, we simply integrate the function from a to b.

In this problem, the function is $y = e^{3x}$, $a = 1$, and $b = 4$.

Note that $e^x > 0$ for all x. So $e^{3x} > 0$ for all x. It follows that the graph of $y = e^{3x}$ lies entirely above the x-axis.

(2) Although it is not needed in this problem, here is a sketch of the area we are being asked to find.



(3) To evaluate $\int e^{3x}dx$, we can formally make the substitution $u = 3x$. It then follows that $du = 3dx$.

We place the 3 next to dx where it is needed, and we leave the $\frac{1}{3}$ outside of the integral sign as follows:

$$\int e^{3x}dx = \frac{1}{3}\int e^{3x} \cdot 3dx$$

We now have

$$\int e^{3x}dx = \frac{1}{3}\int e^{3x} \cdot 3dx = \frac{1}{3}\int e^{u}du = \frac{1}{3}e^{u} + C = \frac{1}{3}e^{3x} + C.$$

We get the second equality by replacing $3x$ by u, and $3dx$ by du.

We get the third equality by the basic integration formula

$$\int e^{u}du = e^{u} + C.$$

And we get the rightmost equality by replacing u with $3x$.

(4) With a little practice, we can evaluate an integral like this very quickly with the following reasoning: The derivative of $3x$ is 3. So we artificially insert a factor of 3 next to dx, and $\frac{1}{3}$ outside the integral sign. Now to integrate $3e^{3x}$ we simply pretend we are integrating e^{x} but as we do it we leave the $3x$ where it is. This is essentially what was done in the above solution.

Note that the 3 "goes away" because it is the derivative of $3x$. We need it to be there for everything to work.

(5) If we are doing the substitution formally, we can save some time by changing the limits of integration. We do this as follows:

$$\int_1^4 e^{3x}\,dx = \frac{1}{3}\int_1^4 e^{3x}\cdot 3dx = \frac{1}{3}\int_3^{12} e^u\,du = \frac{1}{3}e^u\Big|_3^{12} = \frac{1}{3}e^{12} - \frac{1}{3}e^3.$$

Notice that the limits 1 and 4 were changed to the limits 3 and 12. We made this change using the formula that we chose for the substitution: $u = 3x$. When $x = 1$, we have $u = 3(1) = 3$. And when $x = 4$, we have $u = 3(4) = 12$.

Note that this method has the advantage that we do not have to change back to a function of x at the end.

18. Which of the following integrals gives the length of the graph of $y = e^{3x}$ between $x = 1$ and $x = 2$?

(A) $\int_1^2 \sqrt{e^{6x} + e^{3x}}\,dx$

(B) $\int_1^2 \sqrt{x + 3e^{3x}}\,dx$

(C) $\int_1^2 \sqrt{1 + 3e^{3x}}\,dx$

(D) $\int_1^2 \sqrt{1 + 9e^{6x}}\,dx$

Solution: $\frac{dy}{dx} = 3e^{3x}$, so that $1 + \left(\frac{dy}{dx}\right)^2 = 1 + 9e^{6x}$. It follows that the desired length is $\int_1^2 \sqrt{1 + 9e^{6x}}\,dx$, choice (D).

Notes: (1) The **arc length** of the differentiable curve with equation $y = f(x)$ from $x = a$ to $x = b$ is

$$\text{Arc length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

(2) By the chain rule, we have $\frac{dy}{dx} = e^{3x}(3) = 3e^{3x}$. See problem 2 for details.

(3) $\left(\frac{dy}{dx}\right)^2 = (3e^{3x})^2 = 3^2(e^{3x})^2 = 9e^{3x\cdot 2} = 9e^{6x}$. See problem 3 for a review of the laws of exponents used here.

19. $\lim\limits_{x \to 7} \frac{2x^2-13x-7}{x-7} =$

(A) ∞
(B) 0
(C) 2
(D) 15

Solution 1: $\lim\limits_{x \to 7} \frac{2x^2-13x-7}{x-7} = \lim\limits_{x \to 7} \frac{(x-7)(2x+1)}{x-7} = \lim\limits_{x \to 7}(2x+1)$

$$= 2(7) + 1 = 15.$$

This is choice (D).

Notes: (1) When we try to substitute 7 in for x we get the **indeterminate form** $\frac{0}{0}$. Here is the computation:

$$\frac{2(7)^2-13(7)-7}{7-7} = \frac{98-91-7}{7-7} = \frac{0}{0}.$$

This means that we cannot use the method of "plugging in the number" to get the answer. So we have to use some other method.

(2) One algebraic "trick" that works in this case is to factor the numerator as $2x^2 - 13x - 7 = (x-7)(2x+1)$. Note that one of the factors is $(x-7)$ which is identical to the factor in the denominator. This will *always* happen when using this "trick." This makes factoring pretty easy in these problems.

(3) **Most important limit theorem:** If $f(x) = g(x)$ for all x in some interval containing $x = c$ *except* possibly at c itself, then we have $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$.

In this problem, our two functions are

$$f(x) = \frac{2x^2-13x-7}{x-7} \text{ and } g(x) = 2x + 1.$$

Note that f and g agree everywhere *except* at $x = 7$. Also note that $f(7)$ is undefined, whereas $g(7) = 15$.

(4) To compute a limit, first try to simply plug in the number. This will only fail when the result is an indeterminate form. The two **basic** indeterminate forms are $\frac{0}{0}$ and $\frac{\infty}{\infty}$ (the more **advanced** ones are $0 \cdot \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0, but these can always be manipulated into one of the two basic forms).

If an indeterminate form results from plugging in the number, then there are two possible options:

Option 1: Use some algebraic manipulations to create a new function that agrees with the original except at the value that is being approached, and then use the limit theorem mentioned in note (3).

This is how we solved the problem above.

Option 2: Try L'Hôpital's rule (see solution 2 below).

Solution 2: We use L'Hôpital's rule to get

$$\lim_{x \to 7} \frac{2x^2 - 13x - 7}{x - 7} = \lim_{x \to 7} \frac{4x - 13}{1} = 4(7) - 13 = 15, \text{ choice (D).}$$

Notes: (1) L'Hôpital's rule says the following: Suppose that

(i) g and k are differentiable on some interval containing c (except possibly at c itself).

(ii) $\lim_{x \to c} g(x) = \lim_{x \to c} k(x) = 0$ or $\lim_{x \to c} g(x) = \lim_{x \to c} k(x) = \pm\infty$

(iii) $\lim_{x \to c} \frac{g'(x)}{k'(x)}$ exists, and

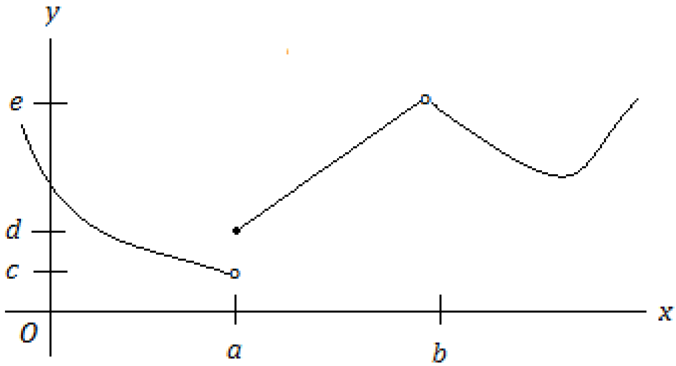
(iv) $k'(x) \neq 0$ for all x in the interval (except possibly at c itself).

Then $\lim_{x \to c} \frac{g(x)}{k(x)} = \lim_{x \to c} \frac{g'(x)}{k'(x)}$.

In this problem $g(x) = 2x^2 - 13x - 7$ and $k(x) = x - 7$.

(2) It is very important that we first check that the expression has the correct form before applying L'Hôpital's rule.

In this problem, note that when we substitute 7 in for x in the given expression we get $\frac{0}{0}$ (see note 1 above). So in this case L'Hôpital's rule can be applied.



20. The graph of the function h is shown in the figure above. Which of the following statements about h is true?

(A) $\lim_{x \to a} h(x) = c$
(B) $\lim_{x \to a} h(x) = d$
(C) $\lim_{x \to b} h(x) = e$
(D) $\lim_{x \to b} h(x) = h(b)$

Solution: From the graph we see that $\lim_{x \to b} h(x) = e$, choice (C).

Notes: (1) The open circles on the graph at a and b indicate that there is *no* point at that location. The darkened circle at a indicates $h(a) = d$.

(2) $\lim_{x \to a^-} h(x) = c$ and $\lim_{x \to a^+} h(x) = d$. Therefore $\lim_{x \to a} h(x)$ does not exist.

(3) h is not defined at $x = b$, ie. $h(b)$ does not exist. In particular, h is not continuous at b. So $\lim_{x \to b} h(x) \neq h(b)$.

21. What is $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\left(\frac{\pi}{4}\right)}{h}$?

(A) 0
(B) 1
(C) 2
(D) The limit does not exist.

Solution 1: If we let $f(x) = \tan x$, then $f'(x) = \lim\limits_{h\to 0} \frac{\tan(x+h)-\tan(x)}{h}$. So $f'\left(\frac{\pi}{4}\right) = \lim\limits_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\left(\frac{\pi}{4}\right)}{h}$.

Now, the derivative of $\tan x$ is $\sec^2 x$. So we have

$$\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\left(\frac{\pi}{4}\right)}{h} = f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2.$$

This is choice (C).

Notes: (1) The derivative of the function f is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

In this problem $f(x) = \tan x$, so that $f'(x) = \lim\limits_{h\to 0} \frac{\tan(x+h)-\tan(x)}{h}$

(2) See problem 1 for the basic trig derivatives. In particular,

$$\frac{d}{dx}[\tan x] = \sec^2 x.$$

(3) $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Therefore $\sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = 1 \div \frac{1}{\sqrt{2}} = 1 \cdot \sqrt{2} = \sqrt{2}$.

(4) $\sec^2\left(\frac{\pi}{4}\right) = \left(\sec\frac{\pi}{4}\right)^2 = \left(\sqrt{2}\right)^2 = 2$.

Solution 2: We use L'Hôpital's rule to get

$$\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{4}+h\right)-\tan\left(\frac{\pi}{4}\right)}{h} = \lim_{h\to 0} \frac{\sec^2\left(\frac{\pi}{4}+h\right)}{1} = \sec^2\left(\frac{\pi}{4}\right) = \left(\sqrt{2}\right)^2 = 2.$$

This is choice (C).

Note: See problem 19 for a detailed description of L'Hôpital's rule.

22. What is $\lim\limits_{x\to\infty} \frac{5-x^2+3x^3}{x^3-2x+3}$?

(A) 1

(B) $\frac{5}{3}$

(C) 3

(D) The limit does not exist.

Solution: $\lim_{x\to\infty} \frac{5-x^2+3x^3}{x^3-2x+3} = \lim_{x\to\infty} \frac{3x^3}{x^3} = 3$, choice (C).

Notes: (1) If p and q are polynomials, then $\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$ where we have $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$.

(2) If $n = m$, then $\lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} = \frac{a_n}{b_m}$.

(3) Combining notes (1) and (2), we could have gotten the answer to this problem immediately by simply taking the coefficients of x^3 in the numerator and denominator and dividing.

The coefficient of x^3 in the numerator is 3, and the coefficient of x^3 in the denominator is 1. So the final answer is $\frac{3}{1} = 3$.

(4) If $n > 0$, then $\lim_{x\to\infty} \frac{1}{x^n} = 0$.

(5) For a more rigorous solution, we can multiply both the numerator and denominator of the fraction by $\frac{1}{x^3}$ to get

$$\frac{5-x^2+3x^3}{x^3-2x+3} = \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} \cdot \frac{(5-x^2+3x^3)}{(x^3-2x+3)} = \frac{\frac{5}{x^3}-\frac{1}{x}+3}{1-\frac{2}{x^2}+\frac{3}{x^3}}.$$

It follows that $\lim_{x\to\infty} \frac{5-x^2+3x^3}{x^3-2x+3} = \frac{5\lim_{x\to\infty}\left(\frac{1}{x^3}\right) - \lim_{x\to\infty}\left(\frac{1}{x}\right) + \lim_{x\to\infty} 3}{\lim_{x\to\infty} 1 - 2\lim_{x\to\infty}\left(\frac{1}{x^2}\right) + 3\lim_{x\to\infty}\left(\frac{1}{x^3}\right)} = \frac{5\cdot 0 - 0 + 3}{1 - 2\cdot 0 + 3\cdot 0} = 3$.

(6) L'Hôpital's rule can also be used to solve this problem since the limit has the form $\frac{\infty}{\infty}$:

$$\lim_{x\to\infty} \frac{5-x^2+3x^3}{x^3-2x+3} = \lim_{x\to\infty} \frac{-2x+9x^2}{3x^2-2} = \lim_{x\to\infty} \frac{-2+18x}{6x} = \lim_{x\to\infty} \frac{18}{6} = 3.$$

Observe that we applied L'Hôpital's rule three times. Each time we differentiated the numerator and denominator with respect to x to get another expression of the form $\frac{\infty}{\infty}$.

See problem 19 for a detailed description of L'Hôpital's rule.

23. $\lim_{h\to 0}\frac{1}{h}\ln(\frac{10+h}{10})$ is equal to

(A) $\frac{1}{10}$

(B) 10

(C) e^{10}

(D) The limit does not exist.

Solution 1: If we let $f(x) = \ln x$, then

$$f'(x) = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{1}{h}\ln(\frac{x+h}{x}).$$

So $f'(10) = \lim_{h\to 0}\frac{1}{h}\ln(\frac{10+h}{10})$.

Now, the derivative of $\ln x$ is $\frac{1}{x}$. So we have

$$\lim_{h\to 0}\frac{1}{h}\ln(\frac{10+h}{10}) = f'(10) = \frac{1}{10}, \text{ choice (A).}$$

Notes: (1) The derivative of the function f is defined by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

In this problem $f(x) = \ln x$, so that $f'(x) = \lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h}$.

(2) Recall that $\ln a - \ln b = \ln\left(\frac{a}{b}\right)$. So $\ln(x+h) - \ln(x) = \ln\left(\frac{x+h}{x}\right)$, and therefore $\frac{\ln(x+h)-\ln(x)}{h} = \frac{1}{h}[\ln(x+h) - \ln(x)] = \frac{1}{h}\ln\left(\frac{x+h}{x}\right)$.

(3) See the notes at the end of problem 3 for a review of the laws of logarithms.

Solution 2: We use L'Hôpital's rule to get

$$\lim_{h\to 0}\frac{1}{h}\ln(\frac{10+h}{10}) = \lim_{h\to 0}\frac{\ln(\frac{10+h}{10})}{h} = \lim_{h\to 0}\frac{\left(\frac{1}{\frac{10+h}{10}}\right)\left(\frac{1}{10}\right)}{1} = \frac{1}{10}, \text{ choice (A).}$$

Note: (1) See problem 19 for more information on L'Hôpital's rule.

(2) To apply L'Hôpital's rule we separately took the derivative of $g(x) = \ln(\frac{10+h}{10})$ and $k(x) = h$.

(3) $g(x) = \ln(\frac{10+h}{10})$ is a composition of the functions $\ln x$ and $\frac{10+h}{10}$. We therefore need to use the Chain Rule to differentiate it.

The first part of the Chain Rule gives us $\frac{1}{\frac{10+h}{10}}$.

For the second part, it may help to rewrite $\frac{10+h}{10}$ as $\frac{1}{10}(10+h)$. It is now easy to see that the derivative of this expression with respect to h is $\frac{1}{10}(0+1) = \frac{1}{10}$.

24. $\lim\limits_{x\to 0} \frac{\sin 7x}{\sin 4x} =$

Solution: $\lim\limits_{x\to 0} \frac{\sin 7x}{\sin 4x} = \lim\limits_{x\to 0} \frac{7\cdot 4x \sin 7x}{4\cdot 7x \sin 4x} = \frac{7}{4} \lim\limits_{x\to 0} \frac{\sin 7x}{7x} \cdot \frac{4x}{\sin 4x}$

$$= \frac{7}{4}(\lim_{7x\to 0} \frac{\sin 7x}{7x})(\lim_{4x\to 0} \frac{4x}{\sin 4x}) = \frac{7}{4}(\lim_{u\to 0} \frac{\sin u}{u}) \frac{1}{(\lim\limits_{v\to 0} \frac{\sin v}{v})} = \frac{7}{4} \cdot 1 \cdot \frac{1}{1} = \mathbf{\frac{7}{4}}.$$

Notes: (1) A basic limit worth memorizing is

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

(2) The limit in note (1) is actually very easy to compute using L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

(3) It is not hard to see that $x \to 0$ if and only if $4x \to 0$ if and only if $7x \to 0$. This is why we can replace x by $4x$ and $7x$ in the subscripts of the limits above.

(4) $\frac{\sin 7x}{\sin 4x}$ can be rewritten as $\frac{7\cdot 4x \sin 7x}{4\cdot 7x \sin 4x}$.

It follows that we can rewrite $\frac{\sin 7x}{\sin 4x}$ as $\frac{7}{4} \cdot \frac{\sin 7x}{7x} \cdot \frac{4x}{\sin 4x}$.

(5) Using the substitution $u = 7x$, we have

$$\lim_{7x\to 0} \frac{\sin 7x}{7x} = \lim_{u\to 0} \frac{\sin u}{u}.$$

Using the substitution $v = 4x$, we have

$$\lim_{4x\to 0} \frac{4x}{\sin 4x} = \frac{1}{\lim\limits_{4x\to 0} \frac{\sin 4x}{4x}} = \frac{1}{\lim\limits_{v\to 0} \frac{\sin v}{v}}$$

(6) We can also solve this problem using L'Hôpital's rule as follows:

$$\lim_{x \to 0} \frac{\sin 7x}{\sin 4x} = \lim_{x \to 0} \frac{7 \cos 7x}{4 \cos 4x} = \frac{7(1)}{4(1)} = \frac{7}{4}.$$

25. $\lim_{x \to 11} \frac{x}{(x-11)^2} =$

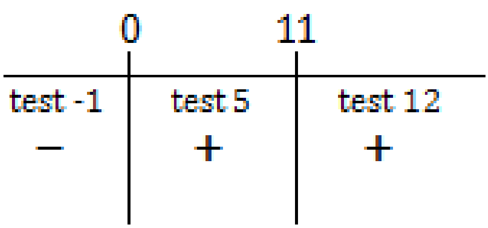
Solution: The function $f(x) = \frac{x}{(x-11)^2}$ has a vertical asymptote of $x = 11$.

If x is "near" 11, then $\frac{x}{(x-11)^2}$ is positive. It follows that $\lim_{x \to 11} \frac{x}{(x-11)^2} = +\infty$.

Notes: (1) When we substitute 11 in for x into $f(x) = \frac{x}{(x-11)^2}$, we get $\frac{11}{0}$. This is *not* an indeterminate form.

For a rational function, the form $\frac{a}{0}$ where a is a nonzero real number *always* indicates that $x = a$ is a vertical asymptote. This means that at least one of $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ is $+$ or $-\infty$. If both limits agree, then $\lim_{x \to a} f(x)$ is the common value. If the two limits disagree, then $\lim_{x \to a} f(x)$ does not exist.

(2) A nice visual way to find the left hand and right hand limits is by creating a **sign chart**. We split up the real line into intervals using the x-values where the numerator and the denominator of the fraction are zero, and then check the sign of the function in each subinterval formed.



In this case we split up the real line into three pieces. Notice that the cutoff points are 0 and 11 because the numerator of the function is zero when $x = 0$, and the denominator of the function is zero when $x = 11$.

We then plug a real number from each of these three intervals into the function to see if the answer is positive or negative. For example, $f(5) = \frac{5}{(5-11)^2} > 0$. Note that we do not need to finish the computation. We only need to know if the answer is positive or negative. Since there are + signs on both sides of $x = 11$, we have that $\lim\limits_{x \to 11} \frac{x}{(x-11)^2} = +\infty$.

(3) We actually do not care about the minus sign to the left of 0. We could have left that part out of the sign chart. It is however important that we include the zero as a cutoff point. This tells us that we can test any value between 0 and 11 to find $\lim\limits_{x \to a^-} f(x)$.

26. Let f be the function defined by

$$f(x) = \begin{cases} \dfrac{5e^{x-7}}{1 + \ln|x-8|}, & x \leq 7 \\ \dfrac{15\cos(x-7)}{\sin(7-x)+3}, & x > 7 \end{cases}$$

Show that f is continuous at $x = 7$.

Solution:

$$\lim_{x \to 7^-} f(x) = \frac{5e^{7-7}}{1+\ln|7-8|} = \frac{5e^0}{1+\ln 1} = \frac{5}{1+0} = 5.$$

$$\lim_{x \to 7^+} f(x) = \frac{15\cos(7-7)}{\sin(7-7)+3} = \frac{15\cos 0}{\sin 0+3} = \frac{15(1)}{0+3} = \frac{15}{3} = 5.$$

So $\lim\limits_{x \to 7} f(x) = 5$.

Also, $f(7) = \frac{5e^{7-7}}{1+\ln|7-8|} = 5$. So, $\lim\limits_{x \to 7} f(x) = f(7)$.

It follows that f is continuous at $x = 7$.

LEVEL 1: SERIES

27. The sum of the infinite geometric series $\frac{5}{7} + \frac{15}{28} + \frac{45}{112} + \cdots$ is

Solution: The first term of the geometric series is $a = \frac{5}{7}$, and the common ratio is $r = \frac{15}{28} \div \frac{5}{7} = \frac{15}{28} \cdot \frac{7}{5} = \frac{3}{4}$. It follows that the sum is

$$\frac{a}{1-r} = \frac{\frac{5}{7}}{1-\frac{3}{4}} = \frac{5}{7} \div \frac{1}{4} = \frac{5}{7} \cdot \frac{4}{1} = \mathbf{\frac{20}{7}}.$$

Notes: (1) A **geometric sequence** is a sequence of numbers such that the quotient $\boldsymbol{r}$ between consecutive terms is constant. The number $\boldsymbol{r}$ is called the **common ratio** of the geometric sequence.

For example, consider the sequence

$$\frac{5}{7}, \frac{15}{28}, \frac{45}{112}, \ldots$$

We have $\frac{15}{28} \div \frac{5}{7} = \frac{15}{28} \cdot \frac{7}{5} = \frac{3}{4}$ and $\frac{45}{112} \div \frac{15}{28} = \frac{45}{112} \cdot \frac{28}{15} = \frac{3}{4}$. It follows that the sequence is geometric with common ratio $r = \frac{3}{4}$.

(2) A **geometric series** is the sum of the terms of a geometric sequence. The series in this problem is an **infinite** geometric series.

(3) The sum G of an infinite geometric series with first term a and common ratio r with $-1 < r < 1$ is

$$\boldsymbol{G} = \frac{\boldsymbol{a}}{\mathbf{1} - \boldsymbol{r}}$$

Note that if the common ratio r is greater than 1 or less than -1, then the geometric series has no sum.

(4) As we saw in note (1), we can get the common ratio r of a geometric series, by dividing any term by the term which precedes it.

28. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{1}{n}$

II. $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+5}$

III. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

(A) I only
(B) II only
(C) III only
(D) I and III only

Solution: The first series is the harmonic series which diverges.

$\lim_{n\to\infty} \frac{n^3}{2n^3+5} = \frac{1}{2}$ and so $\sum_{n=1}^{\infty} \frac{n^3}{2n^3+5}$ diverges by the divergence test.

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Since $\left(\frac{1}{n}\right)$ is a decreasing sequence with $\lim_{n\to\infty} \frac{1}{n} = 0$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test.

So the answer is choice (C).

Notes: (1) $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called the **harmonic series**. This series **diverges**.

It is not at all obvious that this series diverges, and one of the reasons that it is not obvious is because it diverges so slowly.

The advanced student might want to show that given any $M > 0$, there is a positive integer k such that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > M$. This would give a proof that the harmonic series diverges.

(2) The **divergence test** or **nth term test** says:

(i) if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$, or equivalently

(ii) if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Note that statements (i) and (ii) are **contrapositives** of each other, and are therefore **logically equivalent**.

It is usually easier to apply the divergence test by using statement (ii).

In other words, simply check the limit of the underlying *sequence* of the series. If this limit is not zero, then the *series* diverges.

In this problem, the limit of the underlying sequence is $\frac{1}{2}$. Since this is not zero, the given series diverges.

A common mistake is to infer from $\lim_{n\to\infty} \frac{n^3}{2n^3+5} = \frac{1}{2}$ that the series converges to $\frac{1}{2}$. This is of course not true: the *sequence* $\left(\frac{n^3}{2n^3+5}\right)$ converges to $\frac{1}{2}$, but the corresponding *series* diverges by the divergence test.

(3) The **converse** of the divergence test is *false*. In other words, if $\lim_{n\to\infty} a_n = 0$, it does not necessarily follow that $\sum_{n=1}^{\infty} a_n$ converges.

Students make this mistake all the time! It is absolutely necessary for $\lim_{n\to\infty} a_n = 0$ for the series to have any chance of converging. But it is not enough! A simple counterexample is the harmonic series.

To summarize: (a) if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) if $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ may converge or diverge.

(4) To see that $\cos(n\pi) = (-1)^n$, first note that $\cos(0\pi) = \cos 0 = 1$. It follows that $\cos(2k\pi) = \cos(0 + 2k\pi) = 1$ for all integers n, or equivalently, $\cos(n\pi) = 1$ whenever n is even.

Next note that $\cos(1\pi) = \cos\pi = -1$. It then follows that $\cos((2k+1)\pi) = \cos(\pi + 2k\pi) = \cos\pi = -1$ for all integers n, or equivalently, $\cos(n\pi) = -1$ whenever n is odd.

Finally note that $(-1)^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$

(5) An **alternating series** has one of the forms $\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n > 0$ for each positive integer n.

For example, the series given in III is an alternating series since it is equal to $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n}\right)$, and $a_n = \frac{1}{n} > 0$ for all positive integers n.

(6) The **alternating series test** says that if (a_n) is a decreasing sequence with $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Since for all positive integers n, $n < n+1$, it follows that $\frac{1}{n} > \frac{1}{n+1}$, and the sequence $\left(\frac{1}{n}\right)$ is decreasing. Also it is clear that $\lim_{n\to\infty} \frac{1}{n} = 0$. It follows that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test.

(7) Another way to check that the sequence $\left(\frac{1}{n}\right)$ is decreasing is to note that $\frac{d}{dn}\left[\frac{1}{x}\right] = \frac{d}{dn}[x^{-1}] = -1x^{-2} = -\frac{1}{x^2} < 0$. So the function $f(x) = \frac{1}{x}$ is a decreasing function, and therefore the sequence $\left(\frac{1}{n}\right)$ is also decreasing.

29. Let f be a decreasing function with $f(x) \geq 0$ for all positive real numbers x. If $\lim_{b\to\infty} \int_1^b f(x)\,dx$ is finite, then which of the following must be true?

(A) $\sum_{n=1}^{\infty} f(n)$ converges

(B) $\sum_{n=1}^{\infty} f(n)$ diverges

(C) $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges

(D) $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ diverges

Solution: By the integral test, $\sum_{n=1}^{\infty} f(n)$ converges, choice (A).

Notes: (1) The **integral test** says the following:

Let f be a continuous, positive, decreasing function on $[c, \infty)$. Then $\sum_{n=c}^{\infty} f(n)$ converges if and only if $\int_c^{\infty} f(x)\,dx$ converges.

(2) $\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^b f(x)\,dx$.

(3) The integral test cannot be used to evaluate $\sum_{n=1}^{\infty} f(n)$. In general $\sum_{n=1}^{\infty} f(n) \neq \int_1^{\infty} f(x)\,dx$.

(4) The condition of f decreasing can actually be weakened to f "eventually decreasing." For example, $f(x) = \frac{\ln x}{x}$ is not decreasing on $[1, \infty)$, but is decreasing eventually. This can be verified by using the first derivative test. See problem 37 for details on how to apply this test. I leave the details to the reader.

Now, $\int_1^{\infty} \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 \Big|_1^{\infty} = \lim_{b\to\infty} (\ln b)^2 = \infty$. It follows that $\int_1^{\infty} \frac{\ln x}{x}\,dx$ diverges. By the integral test $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

(5) For details on how to integrate $\int \frac{\ln x}{x}\,dx$, see the solution to problem 5.

30. Which of the following series converge to -1 ?

I. $\sum_{n=1}^{\infty} \frac{3}{(-2)^n}$

II. $\sum_{n=1}^{\infty} \frac{1-3n^2}{3n^2+2}$

III. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

(A) I only
(B) II only
(C) III only
(D) I and III only

Solution: The first series is geometric with first term $a = -\frac{3}{2}$ and common ratio $r = -\frac{1}{2}$. So the sum is $\sum_{n=1}^{\infty} \frac{3}{(-2)^n} = \frac{-\frac{3}{2}}{1+\frac{1}{2}} = -1$.

$\lim\limits_{n\to\infty} \frac{1-3n^2}{3n^2+2} = -1$ and so $\sum_{n=1}^{\infty} \frac{1-3n^2}{3n^2+2}$ diverges by the divergence test.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{n\to\infty} [\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)] = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

So the answer is choice (A).

Notes: (1) See problem 27 for more information on infinite geometric series.

(2) For the first series it might help to write out the first few terms:

$$\sum_{n=1}^{\infty} \frac{3}{(-2)^n} = -\frac{3}{2} + \frac{3}{4} - \frac{3}{8} + \cdots + \frac{3}{(-2)^n} + \cdots$$

It is now easy to check that the series is geometric by checking the first two quotients: $\frac{3}{4} \div \left(-\frac{3}{2}\right) = \frac{3}{4} \cdot \left(-\frac{2}{3}\right) = -\frac{1}{2}$, $-\frac{3}{8} \div \frac{3}{4} = -\frac{3}{8} \cdot \frac{4}{3} = -\frac{1}{2}$.

So we see that the series is geometric with common ratio $r = -\frac{1}{2}$. It is also quite clear that the first term is $a = -\frac{3}{2}$.

(3) A geometric series has the form $\sum_{n=0}^{\infty} ar^n$. In this form, the first term is a and the common ratio is r. I wouldn't get too hung up on this form though. Once you recognize that a series is geometric, it's easy enough to just write out the first few terms and find the first term and common ratio as we did in note (2) above.

If we were to put the given series in this precise form it would look like this: $\sum_{n=0}^{\infty}\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)^n$. But again, this is unnecessary (and confusing).

(4) See problem 28 for more information on the divergence test.

(5) The third series is a **telescoping sum**. We can formally do a partial fraction decomposition to see that $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right)$. We start by writing $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$. Now multiply each side of this equation by $n(n+1)$ to get $1 = A(n+1) + Bn = An + A + Bn$.

So we have $0n + 1 = (A+B)n + A$. Equating coefficients gives us $A + B = 0$ and $A = 1$, from which we also get $B = -1$.

So $\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} + \frac{(-1)}{n+1} = \frac{1}{n} - \frac{1}{n+1}$.

(6) Another way to find A and B in the equation $1 = A(n+1) + Bn$ is to substitute in specific values for n. Two good choices are $n = 0$ and $n = -1$.

$n = 0$: $1 = A(0+1) + B(0) = A$. So $A = 1$.

$n = -1$: $1 = A(-1+1) + B(-1)$. So $1 = -B$, and $B = -1$.

31. Which of the following series diverge?

I. $\sum_{n=1}^{\infty}\frac{e^n}{n^2+1}$

II. $\sum_{n=1}^{\infty}\left(\frac{99}{100}\right)^n$

III. $\sum_{n=1}^{\infty}\frac{2^n}{n!}$

(A) I only
(B) II only
(C) III only
(D) I and III only

Solution: $\lim_{n\to\infty}\frac{e^n}{n^2+1}=\infty$ and so $\sum_{n=1}^{\infty}\frac{e^n}{n^2+1}$ diverges by the divergence test.

$\sum_{n=1}^{\infty}(\frac{99}{100})^n$ is geometric with common ratio $r=\frac{99}{100}<1$, and so $\sum_{n=1}^{\infty}(\frac{99}{100})^n$ converges.

$\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right|=\lim_{n\to\infty}\left|\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n}\right|=\lim_{n\to\infty}\frac{2}{n+1}=0<1$, and so $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ converges by the ratio test.

Therefore the answer is choice (A).

Notes: (1) See problems 27 and 30 for more information on infinite geometric series, and see problem 28 for more information on the divergence test.

(2) We say that the series $\sum_{n=0}^{\infty}a_n$ **converges absolutely** if $\sum_{n=0}^{\infty}|a_n|$ converges. If a series converges absolutely, then it converges.

A series which is convergent, but not absolutely convergent is said to **converge conditionally**.

(3) **The Ratio Test:** For the series $\sum_{n=0}^{\infty}a_n$, define $L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

If $L<1$, then the series converges absolutely, and therefore converges. If $L>1$, then the series diverges. If $L=1$, then the ratio test fails.

For the series $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ given in this problem, we have $a_n=\frac{2^n}{n!}$, and so $a_{n+1}=\frac{2^{n+1}}{(n+1)!}$.

32. What are all values of x for which the series $\sum_{n=1}^{\infty}\frac{5^n x^n}{n}$ converges?

(A) All x except $x=0$

(B) $|x|<\frac{1}{5}$

(C) $-\frac{1}{5}\le x<\frac{1}{5}$

(D) $-\frac{1}{5}\le x\le\frac{1}{5}$

Solution: $\lim_{n\to\infty}\left|\frac{\frac{5^{n+1}x^{n+1}}{(n+1)}}{\frac{5^n x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}x^{n+1}}{n+1}\cdot\frac{n}{5^n x^n}\right| = 5|x|$. So by the ratio test, the series converges for all x such that $5|x| < 1$, or equivalently $|x| < \frac{1}{5}$. Removing the absolute values gives $-\frac{1}{5} < x < \frac{1}{5}$.

We still need to check the endpoints. When $x = \frac{1}{5}$, we get the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$, and when $x = -\frac{1}{5}$ we get the convergent alternating series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$. So the series diverges at $x = \frac{1}{5}$ and converges at $x = -\frac{1}{5}$.

The answer is therefore choice (C).

Notes: (1) A **power series** about $x = 0$ is a series of the form $\sum_{n=1}^{\infty} a_n x^n$. To determine where a power series converges we use the ratio test. In other words, we compute

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x|.$$

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