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For Students Currently Scoring Above 600 in SAT Math and Want to Score 800


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SAMPLE

LESSON 1
HEART OF ALGEBRA

In this lesson we will be reviewing four very basic strategies that can be used to solve a wide range of SAT math problems in all topics and all difficulty levels. Throughout this book you should practice using these four strategies whenever it is possible to do so. You should also try to solve each problem in a more straightforward way.

Start with Choice (B) or (C)

In many SAT math problems you can get the answer simply by trying each of the answer choices until you find the one that works. Unless you have some intuition as to what the correct answer might be, then you should always start in the middle with choice (B) or (C) as your first guess (an exception will be detailed in the next strategy below). The reason for this is simple. Answers are usually given in increasing or decreasing order. So very often if choice (C) fails you can eliminate two of the other choices as well.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 2: HEART OF ALGEBRA

$$x - 3 = \sqrt{x + 3}$$

1. What is the solution set of the equation above?

    (A) $\{1\}$
    (B) $\{6\}$
    (C) $\{1,6\}$
    (D) There are no solutions.

See if you can answer this question by starting with choice (B) or (C).

Solution by starting with choice (B): Let's start with choice (B) and guess that the answer is $\{6\}$. We substitute 6 for x into the given equation to get

$$6-3=\sqrt{6+3}$$
$$3=\sqrt{9}$$
$$3=3$$

Since this works, we have eliminated choices (A) and (D). But we still need to check to see if 1 works to decide if the answer is (B) or (C).

Substituting 1 for x into the given equation to get

$$1-3=\sqrt{1+3}$$
$$-2=\sqrt{4}$$
$$-2=2$$

So 1 is not a solution to the given equation and we can eliminate choice (C). The answer is therefore choice (B).

Important note: Once we see that $x=6$ is a solution to the given equation, it is **very important** that we make sure there are no answer choices remaining that also contain 6. In this case answer choice (C) also contains 6 as a solution. We therefore <u>must</u> check if 1 is a solution too. In this case it is not.

Solution by starting with choice (C): Let's start with choice (C) and guess that the answer is {1,6}. We begin by substituting 1 for x into the given equation to get the false equation $-2=2$ (see the previous solution for details). So 1 is not a solution to the given equation and we can eliminate choice (C). Note that we also eliminate choice (A).

Let's try choice (B) now and guess that the answer is {6}. So we substitute 6 for x into the given equation to get the true equation $3=3$ (see the previous solution for details).

Since this works, the answer is in fact choice (B).

Important note: Once we see that $x=6$ is a solution to the given equation, it is **very important** that we make sure there are no answer choices remaining that also contain 6. In this case we have already eliminated choices (A) and (C), and choice (D) does not contain 6 (in fact choice (D) contains no numbers at all).

Before we go on, try to solve this problem algebraically.

Algebraic solution:

$$x - 3 = \sqrt{x + 3}$$
$$(x - 3)^2 = \left(\sqrt{x + 3}\right)^2$$
$$(x - 3)(x - 3) = x + 3$$
$$x^2 - 6x + 9 = x + 3$$
$$x^2 - 7x + 6 = 0$$
$$(x - 1)(x - 6) = 0$$
$$x - 1 = 0 \text{ or } x - 6 = 0$$
$$x = 1 \text{ or } x = 6$$

When solving algebraic equations with square roots we sometimes generate extraneous solutions. We therefore need to check each of the *potential* solutions 1 and 6 back in the original equation. As we have already seen in the previous solutions 6 is a solution, and 1 is not a solution. So the answer is choice (B).

Notes: (1) Do not worry if you are having trouble understanding all the steps of this solution. We will be reviewing the methods used here later in the book.

(2) Squaring both sides of an equation is not necessarily "reversible." For example, when we square each side of the equation $x = 2$, we get the equation $x^2 = 4$. This new equation has two solutions: $x = 2$ and $x = -2$, whereas the original equation had just one solution: $x = 2$.

This is why we need to check for **extraneous solutions** here.

(3) Solving this problem algebraically is just silly. After finding the potential solutions 1 and 6, we still had to check if they actually worked. But if we had just glanced at the answer choices we would have already known that 1 and 6 were the only numbers we needed to check.

When NOT to start with Choice (B) or (C)

If the word **least** appears in the problem, then start with the smallest number as your first guess. Similarly, if the word **greatest** appears in the problem, then start with the largest number as your first guess.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 2: HEART OF ALGEBRA

2. What is the greatest integer x that satisfies the inequality $2 + \frac{x}{5} < 7$?

   (A) 20
   (B) 22
   (C) 24
   (D) 25

See if you can answer this question by starting with choice (A) or (D).

Solution by plugging in answer choices: Since the word "greatest" appears in the problem, let's start with the largest answer choice, choice (D). Now $2 + \frac{25}{5} = 2 + 5 = 7$ This is just barely too big, and so the answer is choice (C).

Before we go on, try to solve this problem algebraically.

*** Algebraic solution:** Let's solve the inequality. We start by subtracting 2 from each side of the given inequality to get $\frac{x}{5} < 5$. We then multiply each side of this inequality by 5 to get $x < 25$. The greatest integer less than 25 is 24, choice (C).

Take a guess

Sometimes the answer choices themselves cannot be substituted in for the unknown or unknowns in the problem. But that does not mean that you cannot guess your own numbers. Try to make as reasonable a guess as possible, but do not over think it. Keep trying until you zero in on the correct value.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 3: HEART OF ALGEBRA

3. Dana has pennies, nickels and dimes in her pocket. The number of dimes she has is three times the number of nickels, and the number of nickels she has is 2 more than the number of pennies. Which of the following could be the total number of coins in Dana's pocket?

   (A) 15
   (B) 16
   (C) 17
   (D) 18

See if you can answer this question by taking guesses.

*** Solution by taking a guess:** Let's take a guess and say that Dana has 3 pennies. It follows that she has $3 + 2 = 5$ nickels, and $(3)(5) = 15$ dimes. So the total number of coins is $3 + 5 + 15 = 23$. This is too many. So let's guess that Dana has 2 pennies. Then she has $2 + 2 = 4$ nickels, and she has $(3)(4) = 12$ dimes for a total of $2 + 4 + 12 = 18$ coins. Thus, the answer is choice (D).

Before we go on, try to solve this problem the way you might do it in school.

Attempt at an algebraic solution: If we let x represent the number of pennies, then the number of nickels is $x + 2$, and the number of dimes is $3(x + 2)$. Thus, the total number of coins is

$$x + (x + 2) + 3(x + 2) = x + x + 2 + 3x + 6 = 5x + 8.$$

So some possible totals are 13, 18, 23,… which we get by substituting 1, 2, 3,… for x. Substituting 2 in for x gives 18 which is answer choice (D).

Warning: Many students incorrectly interpret "three times the number of nickels" as $3x + 2$. This is not right. The number of nickels is $x + 2$, and so "three times the number of nickels" is $3(x + 2) = 3x + 6$.

Pick a number

A problem may become much easier to understand and to solve by substituting a specific number in for a variable. Just make sure that you choose a number that satisfies the given conditions.

Here are some guidelines when picking numbers.

(1) Pick a number that is simple but not too simple. In general you might want to avoid picking 0 or 1 (but 2 is usually a good choice).
(2) Try to avoid picking numbers that appear in the problem.
(3) When picking two or more numbers try to make them all different.
(4) Most of the time picking numbers only allows you to eliminate answer choices. So do not just choose the first answer choice that comes out to the correct answer. If multiple answers come out correct you need to pick a new number and start again. But you only have to check the answer choices that have not yet been eliminated.
(5) If there are fractions in the question a good choice might be the least common denominator (lcd) or a multiple of the lcd.
(6) In percent problems choose the number 100.
(7) Do not pick a negative number as a possible answer to a grid-in question. This is a waste of time since you cannot grid a negative number.
(8) If your first attempt does not eliminate 4 of the 5 choices, try to choose a number that's of a different "type." Here are some examples of types:
    (a) A positive integer greater than 1.
    (b) A positive fraction (or decimal) between 0 and 1.
    (c) A negative integer less than -1.
    (d) A negative fraction (or decimal) between -1 and 0.
(9) If you are picking pairs of numbers try different combinations from (8). For example you can try two positive integers greater than 1, two negative integers less than -1, or one positive and one negative integer, etc.

Remember that these are just guidelines and there may be rare occasions where you might break these rules. For example sometimes it is so quick and easy to plug in 0 and/or 1 that you might do this even though only some of the answer choices get eliminated.

Try to answer the following question using this strategy. **Do not** check the solution until you have attempted this question yourself.

LEVEL 3: HEART OF ALGEBRA

$$\frac{x+y}{x}=\frac{2}{9}$$

4. If the equation shown above is true, which of the following must also be true?

(A) $\frac{x}{y}=\frac{9}{11}$

(B) $\frac{x}{y}=-\frac{9}{7}$

(C) $\frac{x-y}{x}=\frac{11}{9}$

(D) $\frac{x-y}{x}=-\frac{9}{7}$

See if you can answer this question by picking numbers.

Solution by picking numbers: Let's choose values for x and y, say $x=9$ and $y=-7$. Notice that we chose these values to make the given equation true.

Now let's check if each answer choice is true or false.

(A) $\frac{9}{-7}=\frac{9}{11}$ False

(B) $\frac{9}{-7}=-\frac{9}{7}$ True

(C) $\frac{9-(-7)}{9}=\frac{11}{9}$ or $\frac{16}{9}=\frac{11}{9}$ False

(D) $\frac{9-(-7)}{9}=-\frac{9}{7}$ or $\frac{16}{9}=-\frac{9}{7}$ False

Since (A), (C), and (D) are each False we can eliminate them. Thus, the answer is choice (B).

Before we go on, try to solve this problem the way you might do it in school.

Algebraic solution 1: $\frac{x+y}{x}=\frac{x}{x}+\frac{y}{x}=1+\frac{y}{x}$. So the given equation is equivalent to $1+\frac{y}{x}=\frac{2}{9}$. Therefore $\frac{y}{x}=\frac{2}{9}-1=\frac{2}{9}-\frac{9}{9}=-\frac{7}{9}$, and so $\frac{x}{y}=-\frac{9}{7}$, choice (B).

Note: Most students have no trouble at all adding two fractions with the same denominator. For example,

$$\frac{x}{x}+\frac{y}{x}=\frac{x+y}{x}$$

But these same students have trouble reversing this process.

$$\frac{x+y}{x}=\frac{x}{x}+\frac{y}{x}$$

Note that these two equations are **identical** except that the left and right hand sides have been switched. Note also that to break a fraction into two (or more) pieces, the original denominator is repeated for **each** piece.

Algebraic solution 2: We cross multiply the given equation to get

$$9(x+y)=2x$$

We now distribute the 9 on the left to get

$$9x+9y=2x$$

Now we subtract $2x$ from each side of this last equation to get

$$7x+9y=0$$

We subtract $9y$ from each side to get $7x=-9y$.

We can get $\frac{x}{y}$ to one side by performing **cross division.** We do this just like cross multiplication, but we divide instead. Dividing each side of the equation by $7y$ will do the trick (this way we get rid of 7 on the left and y on the right).

$$\frac{x}{y}=\frac{-9}{7}=-\frac{9}{7}$$

This is choice (B).

You're doing great! Let's just practice a bit more. Try to solve each of the following problems by using one of the four strategies you just learned. Then, if possible, solve each problem another way. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

LEVEL 1: HEART OF ALGEBRA

5. If $3z = \frac{y-5}{2}$ and $z = 5$, what is the value of y ?

(A) 20
(B) 25
(C) 30
(D) 35

6. If $x > 0$ and $x^4 - 16 = 0$, what is the value of x ?

LEVEL 2: HEART OF ALGEBRA

7. If $\frac{5x}{y} = 10$, what is the value of $\frac{8y}{x}$?

(A) 4
(B) 3
(C) 2
(D) 1

LEVEL 3: HEART OF ALGEBRA

8. The cost of 5 scarves is d dollars. At this rate, what is the cost, in dollars of 45 scarves?

(A) $\frac{9d}{5}$

(B) $\frac{d}{45}$

(C) $\frac{45}{d}$

(D) $9d$

9. Bill has cows, pigs and chickens on his farm. The number of chickens he has is four times the number of pigs, and the number of pigs he has is three more than the number of cows. Which of the following could be the total number of these animals?

(A) 28
(B) 27
(C) 26
(D) 25

LEVEL 4: HEART OF ALGEBRA

10. For all real numbers x and y, $|x-y|$ is equivalent to which of the following?

(A) $x+y$
(B) $\sqrt{x-y}$
(C) $(x-y)^2$
(D) $\sqrt{(x-y)^2}$

11. If $k \neq \pm 1$, which of the following is equivalent to $\frac{1}{\frac{1}{k+1}+\frac{1}{k-1}}$.

(A) $2k$
(B) k^2-1
(C) $\frac{k^2-1}{2k}$
(D) $\frac{2k}{k^2-1}$

12. In the real numbers, what is the solution of the equation $4^{x+2}=8^{2x-1}$?

(A) $-\frac{7}{4}$
(B) $-\frac{1}{4}$
(C) $\frac{3}{4}$
(D) $\frac{7}{4}$

Answers

1. B	5. D	9. B
2. C	6. 2	10. D
3. D	7. A	11. C
4. B	8. D	12. D

Note: The full solution for question 9 has been omitted because its solution is very similar to the solution for question 3.

Full Solutions

8.

Solution by picking numbers: Let's choose a value for d, say $d = 10$. So 5 scarves cost 10 dollars, and therefore each scarf costs 2 dollars. It follows that 45 scarves cost $(45)(2) = \mathbf{90}$ dollars. **Put a nice big, dark circle around this number so that you can find it easily later.** We now substitute 10 in for d into **all** four answer choices (we use our calculator if we're allowed to).

(A) 90/5 = 18
(B) 10/45
(C) 45/10 = 4.5
(D) 9*10 = 90

Since (D) is the only choice that has become 90, we conclude that (D) is the answer.

Important note: (D) is **not** the correct answer simply because it is equal to 90. It is correct because all 3 of the other choices are **not** 90.

*** Solution using ratios:** We begin by identifying 2 key words. In this case, such a pair of key words is "scarves" and "dollars."

scarves	5	45
dollars	d	x

Notice that we wrote in the number of scarves next to the word scarves, and the cost of the scarves next to the word dollars. Also notice that the cost for 5 scarves is written under the number 5, and the (unknown) cost for 45 scarves is written under the 45. Now draw in the division symbols and equal sign, cross multiply and divide the corresponding ratio to find the unknown quantity x.

$$\frac{5}{d} = \frac{45}{x}$$
$$5x = 45d$$
$$x = 9d$$

So 45 scarves cost $9d$ dollars, choice (D).

10.

Solution by picking numbers: Let's choose values for x and y, let's say $x = 2$ and $y = 5$. Then $|x - y| = |2 - 5| = |-3| = \mathbf{3}$.

Put a nice big dark circle around **3** so you can find it easily later. We now substitute $x = 2$ and $y = 5$ into each answer choice:

(A) 7
(B) $\sqrt{-3}$
(C) $(-3)^2 = 9$
(D) $\sqrt{(-3)^2} = \sqrt{9} = 3$

Since A, B and C each came out incorrect, we can eliminate them. Therefore the answer is choice (D).

*** Solution using the definition of absolute value:** One definition of the absolute value of x is $|x| = \sqrt{x^2}$. So $|x - y| = \sqrt{(x-y)^2}$, choice (D).

Note: Here we have simply replaced x by $x - y$ on both sides of the equation $|x| = \sqrt{x^2}$.

11.

Solution by picking a number: Let's choose a value for k, say $k = 2$. Then

$$\frac{1}{\frac{1}{k+1} + \frac{1}{k-1}} = \frac{1}{\frac{1}{2+1} + \frac{1}{2-1}} = \frac{1}{\frac{1}{3} + 1} = \frac{1}{\frac{1}{3} + \frac{3}{3}} = \frac{1}{\frac{4}{3}} = \mathbf{\frac{3}{4}}$$

Put a nice big, dark circle around this number so that you can find it easily later. We now substitute 2 in for k into **all** four answer choices (we use our calculator if we're allowed to).

(A) $2 * 2 = 4$
(B) $2\text{^}2 - 1 = 3$
(C) $(2\text{^}2 - 1)/(2 * 2) = 3/4$
(D) $(2 * 2)/(2\text{^}2 - 1) = 4/3$

Since (C) is the only choice that has become $\frac{3}{4}$, we conclude that (C) is the answer.

Important note: (C) is **not** the correct answer simply because it is equal to $\frac{3}{4}$. It is correct because all 3 of the other choices are **not** $\frac{3}{4}$.

***Algebraic solution:** We multiply the numerator and denominator of the complex fraction by $(k+1)(k-1)$ to get

$$\frac{1}{\frac{1}{k+1}+\frac{1}{k-1}} \cdot \frac{(k+1)(k-1)}{(k+1)(k-1)} = \frac{(k+1)(k-1)}{(k-1)+(k+1)} = \frac{k^2-1}{2k}$$

This is choice (C).

Notes: (1) The three simple fractions within this complex fraction are $1 = \frac{1}{1}$, $\frac{1}{k+1}$, and $\frac{1}{k-1}$.

The least common denominator (LCD) of these three fractions is

$$(k+1)(k-1)$$

Note that the least common denominator is just the least common multiple (LCM) of the three denominators. In this problem the LCD is the same as the product of the denominators.

(2) To simplify a complex fraction we multiply each of the numerator and denominator of the fraction by the LCD of all the simple fractions that appear.

(3) Make sure to use the distributive property correctly here.

$$\left(\frac{1}{k+1}+\frac{1}{k-1}\right) \cdot (k+1)(k-1)$$
$$= \left(\frac{1}{k+1}\right) \cdot (k+1)(k-1) + \left(\frac{1}{k-1}\right) \cdot (k+1)(k-1)$$
$$= (k-1)+(k+1)$$

This is how we got the denominator in the second expression in the solution.

(4) Do not worry too much if you are having trouble understanding all the steps of this solution. We will be reviewing the methods used here later in the book.

12.

Solution by starting with choice (C) and using our calculator: Let's start with choice (C) and guess that $x = \frac{3}{4}$. We type in our calculator:

$$4\hat{}(3/4+2) \approx 45.255 \quad \text{and} \quad 8\hat{}(2*3/4-1) \approx 2.828$$

Since these two numbers are different we can eliminate choice (C).

Let's try choice (D) next:

$$4\hat{}(7/4+2) \approx 181.019 \quad \text{and} \quad 8\hat{}(2*7/4-1) \approx 181.019$$

Since they came out the same, the answer is choice (D).

*** Algebraic solution:** The numbers 4 and 8 have a common base of 2. In fact, $4 = 2^2$ and $8 = 2^3$. So we have $4^{x+2} = (2^2)^{x+2} = 2^{2x+4}$ and we have $8^{2x-1} = (2^3)^{2x-1} = 2^{6x-3}$. Thus, $2^{2x+4} = 2^{6x-3}$, and so $2x + 4 = 6x - 3$. We subtract $2x$ from each side of this equation to get $4 = 4x - 3$. We now add 3 to each side of this last equation to get $7 = 4x$. Finally we divide each side of this equation by 4 to get $\frac{7}{4} = x$, choice (D).

Note: For a review of the laws of exponents see lesson 13.

LESSON 15
PASSPORT TO ADVANCED MATH

Reminder: Before beginning this lesson remember to redo the problems from Lessons 3, 7 and 11 that you have marked off. Do not "unmark" a question unless you get it correct.

The Distributive Property

The **distributive property** says that for all real numbers a, b, and c

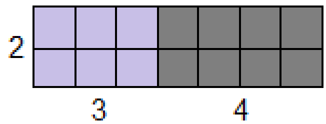
$$a(b+c) = ab + ac$$

More specifically, this property says that the operation of multiplication distributes over addition. The distributive property is very important as it allows us to multiply and factor algebraic expressions.

Numeric example: Show that $2(3+4) = 2 \cdot 3 + 2 \cdot 4$

Solution: $2(3+4) = 2 \cdot 7 = 14$ and $2 \cdot 3 + 2 \cdot 4 = 6 + 8 = 14$.

Geometric Justification: The following picture gives a physical representation of the distributive property for this example.



Note that the area of the light grey rectangle is $2 \cdot 3$, the area of the dark grey rectangle is $2 \cdot 4$, and the area of the whole rectangle is $2(3+4)$.

Algebraic examples: Use the distributive property to write each algebraic expression in an equivalent form.

(1) $2(x+1)$ (2) $x(y-3)$ (3) $-(x-y)$

Solutions: (1) $2(x+1) = 2x + 2$

(2) $x(y-3) = xy - 3x$

(3) $-(x-y) = -x + y$

Factoring

When we use the distributive property in the opposite direction, we usually call it **factoring**.

Examples: (1) $2x + 4y = 2(x + 2y)$

(2) $3x + 5xy = x(3 + 5y)$

(3) $6xy + 9yz = 3y(2x + 3z)$

Here are some more sophisticated techniques for factoring:

The Difference of Two Squares: $a^2 - b^2 = (a - b)(a + b)$

Examples: (1) $x^2 - 9 = (x - 3)(x + 3)$
(2) $4x^2 - 25y^2 = (2x - 5y)(2x + 5y)$
(3) $36 - 49x^2y^2 = (6 - 7xy)(6 + 7xy)$

Trinomial Factoring: $x^2 - (a + b)x + ab = (x - a)(x - b)$

Examples: (1) $x^2 - 5x + 6 = (x - 2)(x - 3)$

(2) $x^2 - 2x - 35 = (x - 7)(x + 5)$

(3) $x^2 + 14x + 33 = (x + 3)(x + 11)$

Square Root Property

The **square root property** says that if $x^2 = a^2$, then $x = \pm a$.

For example, the equation $x^2 = 9$ has the two solutions $x = 3$ and $x = -3$.

Important note: Using the square root property is different from taking a square root. We apply the square root property to an equation of the form $x^2 = a^2$ to get two solutions, whereas when we take the positive square root of a number we get just one answer.

For example when we take the positive square root of 9 we get 3, i.e. $\sqrt{9} = 3$. But when we apply the square root property to the equation $x^2 = 9$, we have seen that we get the two solutions $x = 3$ and $x = -3$.

Example: Solve the equation $(x - 3)^2 = 2$ using the square root property.

Solution: When we apply the square root property we get $x - 3 = \pm\sqrt{2}$. We then add 3 to each side of this last equation to get the two solutions $x = 3 \pm \sqrt{2}$.

Completing the Square

Completing the square is a technique with many useful applications. We complete the square on an expression of the form

$$x^2 + bx$$

To complete the square we simply take half of b, and then square the result. In other words we get $\left(\frac{b}{2}\right)^2$.

The expression $x^2 + bx + \left(\frac{b}{2}\right)^2$ is always a perfect square. In fact,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

For example, let's complete the square in the expression $x^2 + 6x$.

Well half of 6 is 3, and when we square 3 we get 9. So the new expression is $x^2 + 6x + 9$ which factors as $(x + 3)^2$.

Important notes: (1) When we complete the square we usually get an expression that is not equal to the original expression. For example, $x^2 + 6x \neq x^2 + 6x + 9$.

(2) The coefficient of x^2 must be 1 before we complete the square. So, for example, we cannot complete the square on the expression $2x^2 + 32x$.

But we can first factor out the 2 to get $2(x^2 + 16x)$, and then complete the square on the expression $x^2 + 16$ to get $2(x^2 + 16 + 64)$.

Note that we increased the expression by $2 \cdot 64 = 128$.

We will see many applications of completing the square below.

Solving Quadratic Equations

A quadratic equation has the form $\boldsymbol{ax^2 + bx + c = 0}$.

Let's use a simple example to illustrate the various methods for solving such an equation

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$$x^2 - 2x = 15$$

1. In the quadratic equation above, find the positive solution for x.

Solution by guessing: We plug in guesses for x until we find the answer. For example, if we guess that $x = 3$, the we get $3^2 - 2 \cdot 3 = 9 - 6 = 3$. This is too small.

Let's try $x = 5$ next. We get $5^2 - 2 \cdot 5 = 25 - 10 = 15$. This is correct. So the answer is **5**.

Solution by factoring: We bring everything to the left hand side of the equation to get $x^2 - 2x - 15 = 0$. We then factor the left hand side to get $(x - 5)(x + 3) = 0$. So $x - 5 = 0$ or $x + 3 = 0$. It follows that $x = 5$ or $x = -3$. Since we want the positive solution for x, the answer is **5**.

Solution by using the quadratic formula: As in the last solution we bring everything to the left hand side of the equation to get

$$x^2 - 2x - 15 = 0.$$

We identify $a = 1$, $b = -2$, and $c = -15$.

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 + 60}}{2} = \frac{2 \pm \sqrt{64}}{2} = \frac{2 \pm 8}{2} = 1 \pm 4.}$$

So we get $x = 1 + 4 = 5$ or $x = 1 - 4 = -3$. Since we want the positive solution for x, the answer is **5**.

Solution by completing the square: For this solution we leave the constant on the right hand side: $x^2 - 2x = 15$.

We take half of -2, which is -1, and square this number to get 1. We then add 1 to each side of the equation to get $x^2 - 2x + 1 = 15 + 1$. This is equivalent to $(x - 1)^2 = 16$. We now apply the square root property to get $x - 1 = \pm 4$. So $x = 1 \pm 4$. This yields the two solutions $1 + 4 = 5$, and $1 - 4 = -3$. Since we want the positive solution for x, the answer is **5**.

Graphical solution: In your graphing calculator press the Y= button, and enter the following.

Y1 = X^2 – 2X – 15

Now press ZOOM 6 to graph the parabola in a standard window. Then press 2nd TRACE (which is CALC) 2 (or select ZERO), move the cursor just to the left of the second x-intercept and press ENTER. Now move the cursor just to the right of the second x-intercept and press ENTER again. Press ENTER once more, and you will see that the x-coordinate of the second x-intercept is $\mathbf{5}$.

Standard Form for the Equation of a Circle

The standard form for the equation of a circle with center (h, k) and radius r is

$$(\boldsymbol{x} - \boldsymbol{h})^2 + (\boldsymbol{y} - \boldsymbol{k})^2 = \boldsymbol{r}^2.$$

Example: Find the center and radius of the circle with equation $(x - 1)^2 + (y + 2)^2 = 3$

We have $h = 1$ and $k = -2$. So the center of the circle is $(\mathbf{1}, -\mathbf{2})$. The radius is $r = \sqrt{\mathbf{3}}$.

Remark: Note that in this example $(y + 2) = (y - (-2))$. This is why $k = -2$ instead of 2.

General Form for the Equation of a Circle

The general form for the equation of a circle is

$$\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{c} = \mathbf{0}.$$

This form for the equation is not very useful since we cannot easily determine the center or radius of the circle. We will want to apply the method of completing the square twice in order to change the equation into standard form. Let's use an example to illustrate this procedure.

LEVEL 4: ADVANCED MATH

2. In the standard (x, y) coordinate plane, what are the coordinates of the center of the circle whose equation is

$$x^2 - 8x + y^2 + 10y + 15 = 0 \ ?$$

(A) $(4,5)$
(B) $(4, -5)$
(C) $(-4,5)$
(D) $(-5, -4)$

*** Solution by completing the square:**

$x^2 - 8x = x^2 - 8x + 16 - 16 = (x - 4)^2 - 16.$

$y^2 + 10y = y^2 + 10y + 25 - 25 = (y + 5)^2 - 25.$

So $x^2 - 8x + y^2 + 10y + 15 = (x - 4)^2 - 16 + (y + 5)^2 - 25 + 15$
$= (x - 4)^2 + (y + 5)^2 - 26.$

So the center of the circle is $(4, -5)$, choice (B).

Notes: (1) To complete the square in the expression $x^2 - 8x$, we first take half of -8 to get -4. We then square this result to get 16. Note that $x^2 - 8x + 16 = (x - 4)(x - 4) = (x - 4)^2$.

But be aware that it is not really okay to add 16 here – this changes the expression. So we have to undo the damage we just did. We undo this damage by subtracting 16.

(2) To complete the square in the expression $y^2 + 10y$, we first take half of 10 to get 5. We then square this result to get 25. Note that we have $y^2 + 10y + 25 = (y + 5)(y + 5) = (y + 5)^2$.

But be aware that it is not really okay to add 25 here – this changes the expression. So we have to undo the damage we just did. We undo this damage by subtracting 25.

(3) Note that we never finished writing the equation of the circle. We didn't need to since the question asked only to find the center of the circle.

For completeness let's write the equation of the circle. We have

$$(x-4)^2+(y+5)^2-26=0,$$

or equivalently

$$(x-4)^2+(y+5)^2=26$$

So we have an equation of a circle with center $(4,-5)$ and radius $\sqrt{26}$.

Now try to solve each of the following problems. The answers to these problems, followed by full solutions are at the end of this lesson. **Do not** look at the answers until you have attempted these problems yourself. Please remember to mark off any problems you get wrong.

LEVEL 2: ADVANCED MATH

$$5(3x-2)(2x+1)$$

3. Which of the following is equivalent to the expression above?

(A) $30x^2-10$
(B) $30x^2-5x-10$
(C) $25x^2-20$
(D) $15x$

LEVEL 3: ADVANCED MATH

4. In the xy-plane, the parabola with equation $y=(x+7)^2$ intersects the line with equation $y=9$ at two points, P and Q. What is the length of $\overline{PQ}$?

LEVEL 4: ADVANCED MATH

$$h(x)=(x-3)(x+7)$$

5. Which of the following is an equivalent form of the function h above in which the minimum value of h appears as a coefficient or constant?

(A) $h(x)=x^2-21$
(B) $h(x)=x^2+4x-21$
(C) $h(x)=(x-2)^2-21$
(D) $h(x)=(x+2)^2-25$

$$y = cx^2 - k$$
$$y = 5$$

6. In the system of equations above, c and k are constants. For which of the following values of c and k does the system of equations have no real solutions?

(A) $c = -2, k = -6$
(B) $c = 2, k = -6$
(C) $c = 2, k = -4$
(D) $c = 2, k = 4$

$$g(t) = \frac{1}{(t+1)^2 - 6(t+1) + 9}$$

7. For what value of t is the function g above undefined?

LEVEL 5: ADVANCED MATH

8. What are the solutions to $5x^2 - 30x + 20 = 0$?

(A) $x = -20 \pm 20\sqrt{5}$
(B) $x = -20 \pm \sqrt{5}$
(C) $x = 3 \pm 20\sqrt{5}$
(D) $x = 3 \pm \sqrt{5}$

$$y = p(x+3)(x-5)$$

9. In the quadratic equation above, p is a nonzero constant. The graph of the equation in the xy-plane is a parabola with vertex (h, k). What is the value of $h - \frac{k}{p}$?

10. If $(2x + m)(kx + n) = 6x^2 + 29x + c$ for all values of x, and $m + n = 13$, what is the value of c ?

(A) 9
(B) 13
(C) 28
(D) 30

$$x^2 + y^2 - 6x + 2y = -6$$

11. The equation of a circle in the xy-plane is shown above. What is the radius of the circle?

$$x^3 - 3x^2 + 5x - 15 = 0$$

12. For what real value of x is the equation above true?

Answers

1. 5	5. D	9. 17
2. B	6. B	10. D
3. B	7. 2	11. 2
4. 6	8. D	12. 3

Full Solutions

3.

***Solution using the square root property:** Replacing y with 9 in the first equation yields $(x+7)^2 = 9$. We use the square root property to get $x + 7 = \pm 3$. So $x = -7 \pm 3$. So the two solution are $x = -7 + 3 = -4$ and $x = -7 - 3 = -10$.

Sp $P = (-4,9)$ and $Q = (-10,9)$. The distance between these two points is $|-10 - (-4)| = |-10 + 4| = |-6| = \mathbf{6}$.

Notes: (1) To find the points of intersection of the parabola and the line, we solve the given system of equations. We chose to use the **substitution method** here.

(2) Instead of formally applying the square root property to solve $(x+7)^2 = 9$, we can simply "guess" the solutions, or solve the equation informally. It's not too hard to see that $x = -4$ and $x = -10$ will make the equation true.

(3) It's not necessary to write down the points P and Q. Since the y-coordinates of the two points are the same, we can simply subtract one from the other (disregarding the minus sign if it appears) to get the desired distance.

(4) We can also plot the two points and observe that the distance between them is 6

4.

***Quick solution** The x-intercepts of the graph of this function (which is a parabola) are $(3,0)$ and $(-7,0)$. The x-coordinate of the vertex is midway between 3 and -7. So the vertex has x-coordinate $\frac{3-7}{2} = -2$. The answer must therefore be choice (D).

Notes: (1) The minimum value of h is the y-coordinate of the vertex of the parabola that is the graph of h. In this case, the minimum value is $h(-2) = (-2-3)(-2+7) = (-5)(5) = -25$.

(2) The question is really just asking us to rewrite the quadratic function in the standard form $y = a(x-h)^2 + k$. In this form, the minimum value appears as the constant k.

(3) The given equation $h(x) = (x-3)(x+7)$ is in a form where the x-intercepts 3 and -7 of the parabola are displayed as constants.

(Technically an x-intercept is a point and not a number, but the SAT seems to abuse language a bit here, and so I will do the same).

Algebraic solution: We first put the function h into general form by expanding the product $(x-3)(x+7) = x^2 + 4x - 21$.

We now complete the square on $x^2 + 4x$ to get $x^2 + 4x + 4$.

So $x^2 + 4x - 21 = x^2 + 4x + 4 - 4 - 21 = (x+2)^2 - 25$, choice (D).

Important note: The function can also be written $h(x) = x^2 + 4x - 21$ as shown in the algebraic solution. This is answer choice (B). This answer is **wrong** because the minimum value of h, which is -25, does **not** appear as a constant or coefficient!

5.
***Solution using the square root property:** Replacing y with 5 in the first equation yields $5 = cx^2 - k$. Adding k to each side of this equation give us $5 + k = cx^2$. We now divide by c (assuming $c \neq 0$) to get $x^2 = \frac{5+k}{c}$.

We use the square root property to get $x = \pm\sqrt{\frac{5+k}{c}}$.

This will yield no real solutions if the expression under the square root is negative, that is if $5 + k$ and c have opposite signs.

Let's start with choice (C) and guess that $c = 2$ and $k = -4$. Then $5 + k = 1$. Since c and $5 + k$ are both positive, we get 2 real solutions and so we can eliminate choice (C).

Let's try (B) next and guess that $c = 2$ and $k = -6$. Then $5 + k = -1$. So c is positive and $5 + k$ is negative, and the answer is (B).

6.

* g will be undefined when the denominator is zero. So we solve the equation $(t+1)^2 - 6(t+1) + 9 = 0$. The left hand side of the equation factors as $(t+1-3)^2 = 0$, or equivalently $(t-2)^2 = 0$. So $t - 2 = 0$, and therefore $t = \mathbf{2}$.

Note: Many students might find it hard to see how to factor the expression $(t+1)^2 - 6(t+1) + 9$. To help see how to do this we can make a formal substitution of $u = t + 1$. The expression then becomes $u^2 - 6u + 9$ which factors as $(u-3)^2$. The equation $(u-3)^2 = 0$ has solution $u = 3$. But remember that $u = t + 1$. So we have $t + 1 = 3$, and so $t = 3 - 1 = \mathbf{2}$.

7.

* Let's divide through by 5 first to simplify the equation. We get $x^2 - 6x + 4 = 0$. Let's solve this equation by completing the square.

$$x^2 - 6x = -4$$
$$x^2 - 6x + 9 = -4 + 9$$
$$(x-3)^2 = 5$$
$$x - 3 = \pm\sqrt{5}$$
$$\boldsymbol{x = 3 \pm \sqrt{5}}$$

This is choice (D).

Note: This is just one of several ways to solve this problem. See problem 1 in this lesson for several other methods.

8.

* **Solution by completing the square:** Let's put the equation into standard form. We first multiply $(x+3)(x-5)$ to get the equation in general form:

$$y = p(x^2 - 2x - 15)$$

We now complete the square as follows:

$$y = p(x^2 - 2x + 1 - 16) = p(x - 2x + 1) - 16p = p(x-1)^2 - 16p$$

So $h = 1$, $k = -16p$, and $h - \frac{k}{p} = 1 - \frac{-16p}{p} = 1 + 16 = \mathbf{17}$.

9.

* We need $2k = 6$. So $k = 3$. We now have

$$(2x + m)(3x + n) = 6x^2 + 29x + c$$

We need $2n + 3m = 29$.

We can now either use trial and error or formally solve the system of equations

$$3m + 2n = 29$$
$$m + n = 13$$

to get $m = 3$ and $n = 10$

It follows that $c = mn = 3 \cdot 10 = 30$, choice (D).

10.

*** Solution by completing the square:** We put the equation into standard form by completing the square twice:

$$x^2 - 6x + 9 + y^2 + 2y + 1 = -6 + 9 + 1$$
$$(x - 3)^2 + (y + 1)^2 = 4$$

So the radius of the circle is **2**.

11.

*** Solution by factoring:** Let's factor the left hand side of the equation by grouping the first two terms together and the last two terms together.

$$(x^3 - 3x^2) + (5x - 15) = x^2(x - 3) + 5(x - 3) = (x - 3)(x^2 + 5)$$

So we have $(x - 3)(x^2 + 5) = 0$. The only real solution is $x = \mathbf{3}$.

OPTIONAL MATERIAL

LEVEL 6: ADVANCED MATH

1. Let f and g be functions such that $f(x) = ax^2 + bx + c$ and $g(x) = ax + b$. If $g(1) = 2b - a + 25$ and $g(2) = 2a - 24$, then for what value of x does $f(x) = f(8)$, where $x \neq 8$?

Solution

1.

* $g(1) = a(1) + b = a + b$. So $a + b = 2b - a + 25$, and therefore $2a = b + 25$. $g(2) = a(2) + b = 2a + b$, and so $2a + b = 2a - 24$. Thus, $b = -24$. We also have $2a = b + 25 = -24 + 25 = 1$. Thus $a = \frac{1}{2}$.

Then $f(x) = \frac{x^2}{2} - 24x + c$, and $f(8) = \frac{8^2}{2} - 24(8) + c = -160 + c$. If $f(x) = f(8)$, then we have $\frac{x^2}{2} - 24x + c = -160 + c$, and so $\frac{x^2}{2} - 24x + 160 = 0$. Let's multiply each side of this equation by 2 to eliminate the denominator. We get $x^2 - 48x + 320 = 0$. There are several ways to solve this equation. Here are a few.

Factoring: $(x - 8)(x - 40) = 0$. So $x = \mathbf{40}$.

Completing the square: We take half of -48, which is -24, and square this number to get 576. We then add 576 to each side of the equation to get $x^2 - 48x + 576 + 320 = 576$. This is equivalent to $(x - 24)^2 = 256$. We now apply the square root property to get $x - 24 = \pm 16$. So $x = 24 \pm 16$. This yields the two solutions $24 - 16 = 8$, and $24 + 16 = \mathbf{40}$.

The quadratic formula:

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{48 \pm \sqrt{2304 - 1280}}{2} = \frac{48 \pm \sqrt{1024}}{2} = \frac{48 \pm 32}{2} = 24 \pm 16.}$$

As in the previous solution we get $x = 8$ or $x = \mathbf{40}$.

Graphically: In your graphing calculator press the Y= button, and enter the following.

Y1 = X^2 – 48X + 320

Now press ZOOM 6 to graph the parabola in a standard window. It needs to be zoomed out, so we will need to extend the viewing window. Press the WINDOW button, and change Xmax to 100, Ymin to -50, and Ymax to 50. Then press 2nd TRACE (which is CALC) 2 (or select ZERO). Then move the cursor just to the left of the second x-intercept and press ENTER. Now move the cursor just to the right of the second x-intercept and press ENTER again. Press ENTER once more, and you will see that the x-coordinate of the second x-intercept is $\mathbf{40}$.

Remark: The choices made for Xmax, Ymin and Ymax were just to try to ensure that the second x-intercept would appear in the viewing window. Many other windows would work just as well.

About the Author

Steve Warner, a New York native, earned his Ph.D. at Rutgers University in Pure Mathematics in May, 2001. While a graduate student, Dr. Warner won the TA Teaching Excellence Award.

After Rutgers, Dr. Warner joined the Penn State Mathematics Department as an Assistant Professor. In September, 2002, Dr. Warner returned to New York to accept an Assistant Professor position at Hofstra University. By September 2007, Dr. Warner had received tenure and was promoted to Associate Professor. He has taught undergraduate and graduate courses in Precalculus, Calculus, Linear Algebra, Differential Equations, Mathematical Logic, Set Theory and Abstract Algebra.

Over that time, Dr. Warner participated in a five year NSF grant, "The MSTP Project," to study and improve mathematics and science curriculum in poorly performing junior high schools. He also published several articles in scholarly journals, specifically on Mathematical Logic.

Dr. Warner has over 15 years of experience in general math tutoring and over 10 years of experience in AP Calculus tutoring. He has tutored students both individually and in group settings.

In February, 2010 Dr. Warner released his first SAT prep book "The 32 Most Effective SAT Math Strategies." The second edition of this book was released in January, 2011. In February, 2012 Dr. Warner released his second SAT prep book "320 SAT Math Problems arranged by Topic and Difficulty Level." Between September 2012 and January 2013 Dr. Warner released his three book series "28 SAT Math Lessons to Improve Your Score in One Month." In June, 2013 Dr. Warner released the "SAT Prep Official Study Guide Math Companion." In November, 2013 Dr. Warner released the "ACT Prep Red Book – 320 Math Problems With Solutions." Between May 2014 and July 2014 Dr. Warner released "320 SAT Math Subject Test Problems arranged by Topic and Difficulty Level." for the Level 1 and Level 2 tests. In November, 2014 Dr. Warner released "320 AP Calculus AB Problems arranged by Topic and Difficulty Level," and in January, 2015 Dr. Warner released "320 AP Calculus BC Problems arranged by Topic and Difficulty Level,"

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